CATEGORY OF COMPLEMENTED MODULAR LATTICES

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***Abstract*—This paper introduces a category whose objects are comple-mented modular lattices and whose morphisms are normal mappings. Furthermore, we show that complemented modular lattices arising from the submodule lattices of semisimple modules naturally belong to this framework, where module homomorphisms induce linear normal map-pings between the associated lattices. This construction provides a categorical setting for complemented modular lattices and strengthens the connection between lattice theory, module theory, and the theory of normal categories.**

***Index Terms*—Normal category, Complemented modular lattice, Normal cone, Normal mapping, semisimple R-modules**

I. Introduction

The interaction between category theory and algebra has led to significant developments in the structural study of algebraic systems. One of the most influential contributions in this direction is the theory of normal categories, introduced by K.S.S. Nambooripad as a categor-ical framework for the investigation of regular semigroups. A normal category is a category with subobjects satisfying additional structural properties, including the existence of split inclusions, normal factoriza-tions of morphisms, and normal cones. An important feature of this theory is that the collection of all normal cones forms a regular semi-group, thereby establishing a deep connection between categorical and semigroup-theoretic structures.

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Complemented modular lattices constitute a fundamental class of lattices that arise naturally in several branches of algebra, particularly in projective geometry, module theory, and the ideal theory of regular rings. In particular, the lattice of submodules of a semisimple module is a complemented modular lattice, making these lattices an important source of algebraic examples. Their rich structural properties suggest that they provide a suitable setting for the development of categorical methods analogous to those used in the study of regular semigroups.
The notion of a normal mapping between complemented modular lattices, introduced by Nambooripad, provides a natural class of mor-phisms that preserve the order structure while satisfying additional local isomorphism conditions. These mappings capture the essential structural features required for the construction of categorical frame-works and play a role analogous to normal morphisms in the theory of normal categories.
Motivated by these observations, the present paper develops a categorical framework for complemented modular lattices. We define a category, denoted by CModLat, whose objects are complemented modular lattices and whose morphisms are normal mappings. Our main result establishes that this category satisfies all the axioms of a normal category. Specifically, we prove that every inclusion splits, every morphism admits a normal factorization, and each object possesses an associated normal cone. Thus, complemented modular lattices provide another natural class of algebraic structures that admit the theory of normal categories.
As an application of the general construction, we consider comple-mented modular lattices arising from the submodule lattices of semisim-ple modules. We show that module homomorphisms induce linear nor-mal

mappings between these lattices, yielding a natural family of exam-ples within CModLat. This illustrates the close relationship between lattice theory, module theory, and categorical algebra.

II. PRELIMINARIES

We use the standard terminology of category theory through-out. For definitions or results that are not stated explicitly here, the reader may consult the classical references of S. Mac Lane (cf.[8]) and

K.S.S.Nambooripad (cf.[11]). In this paper, all categories under con-sideration are assumed to be small.

A functor is a morphism of categories. Let C and D be two cate-gories. A functor $F : C \to D$ is an embedding(faithful) when to every pair c,c' of objects of C and to every pair $f1, f2 : c \to c'$ of parallel arrows of C the equality $Ff1 = Ff2 : Fc \to Fc'$ implies $f1 = f2$. A functor $F : C \to D$ is an isomorphism of categories when there is a functor $G : D \to C$ such that $FG = IC$ and $GF = ID$.

A preorder P is a category such that, for any $p, p' \in vP$; the hom-set $P(p, p')$ contains at most one morphism. In this case, the relation $\subseteq$ on the class vP defined by $p \subseteq p' \Leftrightarrow P(p, p') \neq \phi$ is a quasi-order on vP . In a preorder, two objects p and p' are isomorphic if and only if $P(p, p') \neq \phi \neq P(p', p)$. Therefore $p \subseteq p'$ is a partial order if and only if P does not contain any nontrivial isomorphisms. Equivalently, the only isomorphisms of P are identity morphisms, and in this case P is said to be a strict preorder.

Let C be a category and P be a subcategory of C. Then (C, P) is called a category with subobjects if the following hold:

1. P is a strict preorder with vP = vC.
2. Every $f \in P$ is a monomorphism in C.
3. If $f, g \in P$ and if $f = hg$ for some $h \in C$, then $h \in P$.

In a category with subobjects, if $f : a \to b$ is a morphism in P, then f is an inclusion and is denoted by $j(a, b)$. If there exists a morphism $e : b \to a$ such that $j(a, b)e = Ia$, then e is called a retraction from $b \to a$, denoted $e(b, a)$. Whenever such a retraction exists, the inclu-sion $j(a, b) : a \to b$ splits.

A morphism $f \in C$, where C be a category with subobjects has factorization if $f = p \cdot m$ where p is an epimorphism and m is an embedding. A category C is said to have the factorisation property if every morphism of C admits such a decomposotion. In this case, each morphism f in C has atleast one canonical factorisation of the form $f = qj$, where q is an epimorphism and j is an inclusion. A normal fac-torisation of a morphism f in C is a factorisation of the form $f = euj$

where e is a retraction, u is an isomorphism and j is an inclusion.

A morphism f in a category with subobjects is said to have an image if it has a canonical(epi-mono)factorization $f = xj$, with x epi and j an inclusion with the property that for any other canonical fac-torisation $f = yj'$, there exists an inclusion j" such that $y = xj''$([11]).

A category is said to have images if every morphism in C has an image. In this case, the codomain of x is said to be the image of f . When the morphism f has an image we denote the unique canonical factorisation of f by $f = fojf$, where fo is the unique epimorphic component and jf is the inclusion of f .

Let C be a category with subobjects, images, every morphism in C has normal factorisations in which the inclusion splits and $d \in vC$. A cone from vC to the vertex d is a map $\gamma : vC \to d$ such that

$\gamma(c) \in C(c, d)$ for all $c \in vC$.

If $c' \subseteq c$ then $j(c', c)\gamma(c) = \gamma(c')$.

The cone γ is called a normal cone if there exists an $a \in vC$ such that $\gamma(a)$ is an isomorphism. The vertex d of the cone γ is usually denoted as $c\gamma$. The set of all normal cones in C is denoted by TC.

Definition 1. A normal category is a pair (C, P) satisfying the follow-ing:

1. (C, P) is a category with subobjects
2. every inclusion in C splits

Any morphism in C has a normal factorization

for each $a \in vC$ there is a normal cone γ with vertex a and

$\gamma(a) = Ia$.

An ideal of a complemented modular lattice L is a sublattice I of L such that $x \le y \in I$ implies $x \in I$. The principal ideal $L(x)$ of L generated by $x \in L$ is $L(x) = \{y \in L | y \le x\}$, which is simply the interval $[0, x]$.

Definition 2. (cf.[12]) Let L and L′ be two complemented modular lattices with bottom and top elements 0, 1 in L and 0′, 1′ in L′. A map $f : L \to L'$ is said to be a normal mapping if it is order preserving, $\mathrm{im} f = [0', a]$ for some $a \in L'$ and $\forall x \in L, \exists z \le x$ such that $f\,|[0,z]$ is an isomorphism from $[0, z]$ onto $[0', f(x)]$.

III. CATEGORY OF COMPLEMENTED MODULAR LATTICES

We now construct a normal category, denoted CModLat, whose objects are complemented modular lattices and whose morphisms are normal mappings.

Theorem 1. The category with complemented modular lattices as ob-jects and morphisms are normal mappings, forms a normal category 'CModLat'.

Proof. In 'CModLat' objects are complemented modular lattices and morphisms are normal mappings between them. For a normal mapping, $f : L \to L'$ the lattice L serves as the domain and L′ as the codomain. Composition of two normal mappings is again a normal mapping and the identity map $IL : L \to L$ is also normal. Since associativity for composition is inherited from funtion composition and $f \cdot IL = f =$ $' \cdot f$, the structure indeed forms a category.

The ideals of the lattices, $[0, a]$ act as subobjects. If $f : L \subseteq L'$ where $L = [0, a]$ and $L' = [0, b]$ where $a \le b$; then there exists a map $e : L' \to L$ such that

$$e(c) = \{c, \text{if } c \;\; L\; 0, \text{if } c \;\; [a, b]$$

showing that every inclusion splits. Moreover, by the third condition of normal mapping, for each $x \in L$, there exists some $b \le x$ such that the restriction $f\,|[0, b] : [0, b] \to [0, f(x)]$ is an isomorphism. Hence every morphism $f : L \to L'$ admits a normal factorization $f = quj$ where $q : L \to [0, b]$ is a retraction, $u : [0, b] \to [0, f(x)]$ is an isomorphism and $j : [0, f(x)] \to L'$ is an inclusion. A normal cone with vertex L is a map $\gamma : vCModLat \to L$ such that

$\gamma(L1) \in \mathrm{hom}(L1, L)$

if $L1 \subseteq L2$, then $j(L1, L2) \cdot \gamma(L2) = \gamma(L1)$

there exists atleast one L3 such that $\gamma(L3)$ is an isomorphism.

There is also an idempotent normal cone with vertex L obtained by taking $\gamma(L) = IL$. Hence 'CModLat' is a normal category.

3.1 SUBMODULES AND COMPLEMENTED MODULAR
LATTICE. We now focus on the complemented modular lattices that arise from the submodule structure of semisimple R-modules and ex-amine the associated normal mappings.
As we know the family of submodules of a semisimple R-module
M forms a complemented modular lattice, denoted L(M). Let $\theta : M \to$
N be an R-module homomorphism between semisimple R-modules
M and N , then by Shur's lemma θ is an isomorphism. For each submodule $Mi \in L(M)$, define $f\theta(Mi) = \theta(Mi)$. This gives a map

$f\theta : L(M) \to L(N)$, which is a normal mapping between the com-plemented modular lattices L(M) and L(N). Indeed if $M1 \leq M2$, then $\theta(M1) \leq \theta(M2) \to f\theta(M1) \leq f\theta(M2)$, so fθ preserves the or-der. The image of fθ is the ideal generated by N . Moreover, for every Mi in L(M), there exists $Mj \leq Mi$ such that the restriction $f\theta|[0,M] : [0, Mj] \to [o, f\theta(Mi)]$ is an isomorphism. Let us call this normal mapping fθ a 'linear normal mapping' as it coincides with R-
module homomorphisms.
Define $hom(L(M), L(N)) = \{f\theta | \theta \in homR(M, N)\}$, where $f\theta(N) = \theta(N)$ and each fθ is order preserving, join preserving and a linear nor-mal mapping. It follows that the complemented modular lattices ob-tained from the submodules of semisimple R-modules together with
these linear normal mappings, form a normal category.

IV. CONCLUDING REMARKS

The categorical framework developed here opens several direc-tions for further investigation. In particular, it would be interesting to study cross-connections associated with complemented modular lat-tices, characterize the corresponding semigroup structures of normal cones, and extend the construction to broader classes of modular lat-tices and algebraic structures such as regular rings and related cate-gories.

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