On the Coincidence point theorem in symmetric intuitionistic Nb- fuzzy metric space

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Abstract—**In the present paper, we define the concept of compatible maps and compatible of type-P maps in symmetric intuitionistic Nb- fuzzy metric space with examples. As an application of compatible type-P maps we have investigated a coincidence point theorem with example.**

Index Terms—**Intuitionistic Nb-fuzzy metric space, coincidence point compatible mapping, compatible mapping of type (P)**

AMS Mathematics Subject Classification—**47H10, 54H25, 54E40**

I. Introduction

The concept of fuzzy metric space was introduced by Kramosil and Michalek [8] in 1975. It was modified by George and Veeramani [5].

On the other hand Gahler [4] Generalized usual notion of metric space that is known as 2- metric space. S. Sharma [15] and S. Kumar [9] also introduced fuzzy -2-metric spaces without knowing each other but Ha et.al.[6] proved that 2-metric need not be continuous function, the results obtained in the two settings had no easy relationship between them. In 1992 Bapure Dhage [3] in his Ph.D. thesis introduced a new class of generalized metric space called the D-metric space. However, Mustafa and Sims [10,11] both have pointed out that most of the D- metric are invalid. To overcome some of these fundamental flaws, they have introduced the notion of G metric space. Sun and K Yang [16] introduced the notion of Q-fuzzy metric space by using G metric space, Sedghi et.al [14] introduced S-metric space which is a generalization of G- metric space. In 2015, N. Malviya [12] introduced the notion *N*-fuzzy metric space using the compatible of S metric space and describe some of their properties and examples. Recently in Ali et al.[2] introduced the notion of Intuitionistic *N*- fuzzy metric space ($I\ N\ F\ M\ S$), very recently in 2023 Ishtiaq et al.[7] defined Intuitionistic N_b- fuzzy metric space ($I\ N_b F\ M\ S$), which is the generalization of fuzzy metric space.

In the this paper, we define compatible and compatible of type (p) maps in intuitionistic N_b-fuzzy metric space($I\ N_b\ F\ M\ S$) and proved coincidence point theorem in symmetric intuitionistic N_b-fuzzy metric space($I\ N_b\ F\ M\ S$) which generalize & extend the theorem of Agrawal S. et al [1] and Tenguria et al.[17]

II. Preliminaries

Definition 2.1 [9] A map $\star: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t- norm if it satisfies the following conditions:

$T_1: \star(\mu, 1, 1) = \mu, \ \star(0,0,0) = 0$

$T_2: \star(\mu, \nu, \rho) = \star(\mu, \rho, \nu) = \star(\nu, \rho, \mu)$

$T_3: \star$ is continuous

$T_4: \star(\mu_1, \nu_{1,} \rho_1) \geq \star(\mu_2, \nu_{2,} \rho_{2,}) \ for\ \mu_1 \geq \mu_2, \nu_{1,} \geq \nu_{2,}, \ \rho_1 \geq \rho_2$

examples of t-norm are (1): $\mu \star \nu \star \rho = \mu.\nu.\rho$ and (2): $\mu \star \nu \star \rho = min\{\mu, \nu, \rho\}(H - type)$

Definition 2.2 [9] A map $\star: [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t- co-norm (CTCN) if it satisfies the following conditions:

T_1:$\circledast(\mu, 0,0) = \mu, \circledast(0,0,0) = 0$

T_2:$\circledast(\mu, \nu, \rho) = \circledast(\mu, \rho, \nu) = \circledast(\nu, \rho, \mu)$

T_3:$\star$ is continuous

T_4:$\circledast(\mu_1, \nu_{1,} \rho_1) \geq \circledast(\mu_2, \nu_{2,} \rho_{2,}) \, for \, \mu_1 \geq \mu_2, \nu_1 \geq \nu_{2,}, \rho_1 \geq \rho_2$

$\mu \circledast \nu \circledast \rho = max\{\mu, \nu, \rho\}$ are called a maximum CTCN.

The concept of intuitionistic N_b -fuzzy metric space is defined as follows:

Definition 2.3[7]:- A six tuple (X, $M, N, \star, \circledast, s$) is an intuitionistic N_b-fuzzy metric space($I\ N_b\ F\ M\ S$), if X is an arbitrary (non-empty) set, $\star$ is a continuous t-norm, $\circledast$ is a continuous t- co-norm (CTCN), $s \geq 1$ is a real number, $M and\ N$ are a fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions for all $\mu, \nu, \rho, \xi \epsilon\ X$ and $t, u, v > 0$:

$(i)\ M(\mu, \nu, \rho, t) + N(\mu, \nu, \rho, t) \leq 1$

$(ii)\ M(\mu, \nu, \rho, t) > 0$

$(iii)\ M(\mu, \nu, \rho, t) = 1\ if\ and\ only\ if\ \mu = \nu = \rho$

$(iv) M(\mu, \nu, \rho, s(u + v + t)) \geq M(\mu, \mu, \xi, u) \star M(\nu, \nu, \xi, v) \star M(\rho, \rho, \xi, t)$ for all $\xi \in X$.

$(v)\ M(\mu, \nu, \rho, .)\colon (0, \infty) \to (0,1]\ is\ a\ continuous\ function.$

$(vi)\ N(\mu, \nu, \rho, t) > 0$

$(vii)\ N(\mu, \nu, \rho, t) = 1\ if\ and\ only\ if\ \mu = \nu = \rho$

$(viii) N(\mu, \nu, \rho, s(u + v + t)) \leq N(\mu, \mu, \xi, u) \circledast N(\nu, \nu, \xi, v) \circledast N(\rho, \rho, \xi, t)$ for all $\xi \in X$.

$(ix)\ N(\mu, \nu, \rho, .)\colon (0, \infty) \to (0,1]\ is\ a\ continuous\ function.$

Here, $M(\mu, \nu, \rho, t)$ and $N(\mu, \nu, \rho, t)$ are membership and non-membership function of μ, $\nu\ and\ \rho\ with\ respect\ to\ t$.

Definition 2.4. [7] Let s ≥ 1 be a given real number. *A* function f: R → R will be called strictly increasing if t < u implies that f (t) ≤ f (su) and f is called strictly decreasing if t < u implies that f (t) ≥ f (su).

Proposition 2.5. [7] Let (X,$M, N, \star, \circledast, s$) is an intuitionistic N_b- fuzzy metric space($I\ N_b\ F\ M\ S$), then for all $\mu, \nu \in$ X, the fuzzy set $M\ and\ N$ are defined with respect to product such that $M(\mu, \mu, \nu\ \star)$: $[0, \infty) \to [0, 1]$ is strictly increasing and $N(\mu, \mu, \nu\ \circledast)$: $[0, \infty) \to [0, 1]$ strictly decreasing.

Definition 2.6.[7] Suppose (X,$M, N, \star, \circledast, s$) is an intuitionistic N_b- fuzzy metric space($I\ N_b\ F\ M\ S$), then (X,$M, N, \star, \circledast, s$) is called symmetric if

$M(\mu, \mu, \nu, t) = M(\nu, \nu, \mu, t)$ and $N(\mu, \mu, \nu, t) = N(\nu, \nu, \mu, t)$.

For all $\mu, \nu \in X$ and t > 0.

Definition 2.7. [7] Let $(X, M, N, \star, \circledast, s)$ is symmetric intuitionistic N_b- fuzzy metric space.

(a) A sequence $\{\mu_n\}$ in X is said to be convergent if there exists $\mu \in X$ such that $\lim_{n\to\infty} M(\mu_n, \mu_n, \mu, t) = 1$ and $\lim_{n\to\infty} N(\mu_n, \mu_n, \mu, t) = 0$ $\forall t > 0$. In this case μ is called the limit of the sequence $\{\mu_n\}$ and we write $\lim_{n\to\infty} \mu_n = \mu$, or $\mu_n \to \mu$.

(b) A sequence $\{\mu_n\}$ in $(X, M, N, \star, \circledast, s)$ is said to be a Cauchy sequence if for every $\epsilon \in (0, 1)$, there exists $n_0 \in N$ such that $M(\mu_n, \mu_n, \mu_m, t) > 1-\epsilon$ and $N(\mu_n, \mu_n, \mu_m, t) < \epsilon$, $\forall m, n \geq n_0$ and $t > 0$.

(c) The space X is said to be complete if every Cauchy sequence is convergent and it is called compact if every sequence has a convergent subsequence.

III. Main Result

We define compatible and compatible of type-P mappings in symmetric intuitionistic N_b -fuzzy metric spaces.

Definition 3.1. Two self-mappings A and B of an symmetric intuitionistic N_b -fuzzy metric space $(X, M, N, \star, \circledast, s)$ are called compatible if $M(AB\mu_n, AB\mu_n, BA\mu_n, t) = 1$ and $N(AB\mu_n, AB\mu_n, BA\mu_n, t) = 0$ whenever $\{\mu_n\}$ is a sequence in X such that $B\mu_n = A\mu_n = \mu$ for all some $\mu \in X$.

Definition 3.2. Two self-mappings A and B of an symmetric intuitionistic N_b -fuzzy metric space $(X, M, N, \star, \circledast, s)$ are called compatible of type (P) if $M(AA\mu_n, AA\mu_n, BB\mu_n, t) = 1$ and $N(AA\mu_n, AA\mu_n, BB\mu_n, t) = 0$ whenever $\{\mu_n\}$ is a sequence in X such that $B\mu_n = A\mu_n = \mu$ for all some $\mu \in X$.

Example 3.2.1. Let $X = \{\frac{1}{n} : n\epsilon N\} \cup \{0\}$ with $\star$ continuous t-norm and $\circledast$ continuous t-conorm defined by $p \star q = pq$ and $p \circledast q = min\{1, p+q\}$ respectively, for $p, q\epsilon[0,1]$. For each $t \in [0, \infty)$ and $\mu, \nu \in X$, define (M, N) by

$$M(\mu, \nu, w, t) = \{\frac{t}{t+|(\mu-w)-(w-\nu)|^2}, if\ \ t > 0, 0, \qquad if\ \ t = 0, \quad \text{and}$$

$$N(\mu, \nu, w, t) = \{\frac{|(\mu-w)-(w-\nu)|^2}{t+|(\mu-w)-(w-\nu)|^2}, if\ \ t > 0, 1, \qquad if\ \ t = 0,$$

Clearly $(X, M, N, \star, \circledast, s)$ is a symmetric intuitionistic N_b -fuzzy metric space.
Define $A\mu = \frac{\mu}{6}$ and $Q\mu = \frac{\mu}{2}$ on X and $\mu_n = \frac{1}{n}$.
Clearly, it can be easily observed that A and Q are compatible of type (P) mapping.

Our main result extends and generalize the fixed-point theorem of Agrawal et al [1], which is the application of compatible of type(P) mapping in the new structure of symmetric intuitionistic N_b -fuzzy metric space.

Theorem 3.3 Let $(X, M, N, \star, \circledast, s)$ be a complete symmetric intuitionistic N_b -fuzzy metric space with $\star$ t-norm and $\circledast$ t-conorm defined as:

1. $\mu \star \nu = \min\{\mu, \nu\}$, $\mu \circledast \nu = \max\{\mu, \nu\}$,
2. $M(\mu, \mu, \nu \star)$ and $N(\mu, \mu, \nu \circledast)$ are strictly increasing and strictly decreasing functions, respectively.

Let $B, A: X \to X$ be two self-mapping on X satisfy following conditions:

(i) $A(X) \subseteq B(X)$
(ii) One of B or A is continuous.
(iii) (B, A) is compatible of type (P)
(iv) If for all $\mu, \nu \in X, k \in \left(0, \frac{1}{3}\right), t > 0$,

$$M(A\mu, A\mu, A\nu, kt) \geq \min\{M(A\mu, A\mu, B\nu, t), M(A\nu, A\nu, B\nu, t), M(A\nu, A\nu, B\mu, t)\}$$

$$N(A\mu, A\mu, A\nu, kt) \leq \max\{N(A\mu, A\mu, B\nu, t), N\ A\nu, A\nu, B\nu, t), N(A\nu, A\nu, B\mu, t)\}$$

Then μ is common fixed point of B and A.

Proof. Let $\mu_0 \in X$. Since $A(X) \subseteq B(X)$ there exist μ_{2n+1} and μ_{2n} in X such that

$R\mu_{2n} = Q\mu_{2n+1} = \nu_{2n+1}$ for, n = 1, 2, 3, ... (3.3.1)

Case I. Putting $\mu = \mu_{2n}$ and $\nu = \mu_{2n+1}$ in (iv) we get

$$M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) = M(B\mu_{2n+1}, B\mu_{2n+1}, B\mu_{2n+2}, kt) = M(A\mu_{2n}, A\mu_{2n}, A\mu_{2n+1}, kt)$$

$$\geq \min\{M(A\mu_{2n}, A\mu_{2n}, B\mu_{2n+1}, t), M(A\mu_{2n+1}, A\mu_{2n+1}, B\mu_{2n+1}, t), M(A\mu_{2n+1}, A\mu_{2n+1}, B\mu_{2n}, t)\},$$

$$= \min\{M(B\mu_{2n+1}, B\mu_{2n+1}, B\mu_{2n+1}, t), M(B\mu_{2n+2}, B\mu_{2n+2}, B\mu_{2n+1}, t), M(B\mu_{2n+2}, B\mu_{2n+2}, B\mu_{2n}, t)\},$$

$$= \min\{M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+1}, t), M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)\},$$

Since $M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+1}, t) = 1$.

$$M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \geq \min\{(1, M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\},$$

$$\geq \min\{(M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\},$$

Since $kt < \frac{t}{3s}$ and by (ii) of theorem 3.3 $M(\mu, \mu, \nu \star)$ is a strictly increasing function.

If $\min\{(M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\},= M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t)$

Then we will reach to a contradiction $M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \geq M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t)$

Therefore,

$M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \geq M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)$

$\geq M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s}) \star M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s}) \star M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3s})$ (By using (iv) of definition 2.3

$= \min\{M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s}), M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3s})\}$ (By (i) of theorem 3.3)

Since $kt < \frac{t}{3s}$ and by (ii) of theorem 3.3. $M(\mu, \mu, \nu \star)$ is a strictly increasing function.

If $\min\{M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+2}, \frac{t}{3s}), M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3s})\} = M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s})$,

Then we will again reach to contradiction,

$M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \geq M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, \frac{t}{3s})$

Which is not possible. Therefore, $M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, kt) \geq M(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3s})$

In the similar manner, $M(\nu_{2n+3}, \nu_{2n+3}, \nu_{2n+2}, kt) \geq M(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s})$

In general, $M(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \geq M(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3s})$ for n=1,2,3,….

And, $M(\nu_{n+2}, \nu_{n+2}, \nu_{n+3}, kt) \geq M(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{3s})$ for n=1,2,3,….

Also, it follows that,

$M\ (\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \geq M\ (\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3s}) \geq M\ (\nu_{n-1}, \nu_{n-1}, \nu_n, \frac{t}{(3s)^2 k})$.

Continuing this, we get,

$M\ (\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \geq M\ (\nu_0, \nu_0, \nu_1, \frac{t}{(3s)^{n+1}k^n}) \to 0$ as $n \to \infty$.

Thus, in general, when $n \to \infty$, clearly,

$1 \geq M\ (\nu_n, \nu_n, \nu_{n+1}, kt) \geq M\ (\nu_0, \nu_0, \nu_1, \frac{t}{(3s)^n k^{n-1}}) \to 1$

Thus, $M\ (\nu_n, \nu_n, \nu_{n+1}, kt) = 1$.

Furthermore,

$N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) = N\ (B\mu_{2n+1}, B\mu_{2n+1}, B\mu_{2n+2}, kt) = N\ (A\mu_{2n}, A\mu_{2n}, A\mu_{2n+1}, kt)$

$\leq \max\{ N\ (A\mu_{2n}, A\mu_{2n}, B\mu_{2n+1}, t), N\ (A\mu_{2n+1}, A\mu_{2n+1}, B\mu_{2n+1}, t), N\ (A\mu_{2n+1}, A\mu_{2n+1}, B\mu_{2n}, t)\}$,

$= \max\{ N\ (B\mu_{2n+1}, B\mu_{2n+1}, B\mu_{2n+1}, t), N\ (B\mu_{2n+2}, B\mu_{2n+2}, B\mu_{2n+1}, t), N\ (B\mu_{2n+2}, B\mu_{2n+2}, B\mu_{2n}, t)\}$,

$= \max\{ N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+1}, t), N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)\}$,

$\Rightarrow N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq max\{N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+}, t), N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)\}$

Since $N(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+1}, t) = 0$.

$N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq \max\{(1,\ N(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\}$,

$\leq \max\ \{(N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\}$,

Since $kt < \frac{t}{3s}$ and by of theorem 3.3 $N(\ \mu, \mu, \nu\ \star)$ is a strictly decreasing function.

If $\max\{(N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\} ,= N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t)$

Then we will reach to a contradiction $N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t)$ is not possible.

Therefore,

$N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)$

$\leq\ N(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s}) * N(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s}) * N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3s})$

$= \max\ \{ N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s}), N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3s})\}$

Since $kt < \frac{t}{3s}$ and by (ii) of theorem 3.3. $N(\ \mu, \mu, \nu\ *)$ is a strictly decreasing function.

If $\max\ \{ N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+2}, \frac{t}{3s}), N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3s})\} = N\ (\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s})$,

Then we will again reach to contradiction,

$N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq\ N\ (\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, \frac{t}{3s})$

Which is not possible. Therefore, $N\,(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, kt) \leq N\,(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3s})$

In the similar manner, $N\,(\nu_{2n+3}, \nu_{2n+3}, \nu_{2n+2}, kt) \leq N\,(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3s})$

In general, $N\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \leq N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s})$ for n=1,2,3,….

And, $N\,(\nu_{n+2}, \nu_{n+2}, \nu_{n+3}, kt) \leq N\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{3s})$ for n=1,2,3,….

Also, it follows that, $N\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \leq N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s}) \leq N\,(\nu_{n-1}, \nu_{n-1}, \nu_{n}, \frac{t}{(3s)^2 k})$.

Continuing this, we get, $N\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \leq N\,(\nu_{0}, \nu_{0}, \nu_{1}, \frac{t}{(3s)^{n+1} k^n}) \rightarrow 0$ as $n \rightarrow \infty$.

Thus, in general, when $n \rightarrow \infty$, clearly,

$0 \leq N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, kt) \leq N\,(\nu_{0}, \nu_{0}, \nu_{1}, \frac{t}{(3s)^{n+1} k^{n-1}}) \rightarrow 0$

Therefore, $N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, kt) = 0$.

Hence, $M\,(\nu_{n}, \nu_{n}, \nu_{n+1}, kt) \rightarrow 1$. And $N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, kt) \rightarrow 0$ as $n \rightarrow \infty$ for any $t > 0$,

Next, we show that the sequence $\{\nu_n\}$ is a Cauchy sequence.

For each ε > 0 and t > 0, we may be chosen $n_0 \in N$ such that

$M\,(\nu_{n}, \nu_{n}, \nu_{n+1}, t) > 1 - \varepsilon$ for all $n > n_0$ and $N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, t) < \varepsilon$ for all $n > n_0$

For m, n ∈ N, we suppose m ≥ n. Then we have

$M\,(\nu_{n}, \nu_{n}, \nu_{m}, t) \geq M\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s}) \star M\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s}) \star M\,(\nu_{n+1}, \nu_{n+1}, \nu_{m}, \frac{t}{3s})$

$\geq M\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s}) \star M\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3s)^2}) \star M\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3s)^2})$

$\star M\,(\nu_{n+2}, \nu_{n+2}, \nu_{m}, \frac{t}{(3s)^2})$

$\geq M\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s}) \star M\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3s)^2}) \star M\,(\nu_{n+2}, \nu_{n+2}, \nu_{n+1}, \frac{t}{(3s)^3}) \ldots$

$\Rightarrow M\,(\nu_{n}, \nu_{n}, \nu_{n+1}, t) \geq (1 - \varepsilon) \star (1 - \varepsilon) \star (1 - \varepsilon) \ldots (1 - \varepsilon)$

$= \min\{(1 - \varepsilon), (1 - \varepsilon), (1 - \varepsilon) \ldots (1 - \varepsilon)\}$

And

$N\,(\nu_{n}, \nu_{n}, \nu_{m}, t) \leq N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s}) \circledast N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s}) \circledast N\,(\nu_{n+1}, \nu_{n+1}, \nu_{m}, \frac{t}{3s})$

$\leq N\,(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3s}) \circledast N\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3s)^2}) \circledast N\,(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3s)^2}) \circledast N\,(\nu_{n+2}, \nu_{n+2}, \nu_{m}, \frac{t}{(3s)^2})$

$\leq N(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3s}) \circledast N(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3s)^2}) \circledast N(\nu_{n+2}, \nu_{n+2}, \nu_m, \frac{t}{(3s)^3}) \ldots$

$\Rightarrow N(\nu_n, \nu_n, \nu_{n+1}, t) \leq \varepsilon \circledast \varepsilon \circledast \varepsilon \circledast \ldots \circledast \varepsilon$

$= \max\{\varepsilon, \varepsilon, \varepsilon, \ldots \varepsilon\} = \varepsilon$

Hence, $\{\nu_n\}$ is a Cauchy sequence in X.

Since $(X, M, N, \star, \circledast, k)$ is complete. In view of completeness of the space, sequence $\{\nu_n\}$ converges to some point $u \in X$. Also, its subsequence converges to the same Point i.e., $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$. Now, shall prove $B\alpha = \alpha$ then

$M(\alpha, \alpha, B\alpha, kt) \geq M(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) \star M(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) \star M(A\mu_{2n}, A\mu_{2n}, B\alpha, \frac{kt}{3s}),$

B is continuous and A, B are Compatible type P

such that $n \to \infty$. $AA\mu_{2n} \to B\alpha, BB\mu_{2n} \to B\alpha,$

$M(\alpha, \alpha, B\alpha, kt) \geq M(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) \star M(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) \star M(A\mu_{2n}, A\mu_{2n}, AA\mu_{2n}, \frac{kt}{3s}),$

$\geq M(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) \star min\{M(A\mu_{2n}, A\mu_{2n}, BA\mu_{2n}, \frac{t}{3s}), M(AA\mu_{2n}, AA\mu_{2n}, BA\mu_{2n}, \frac{t}{3s}),$

$M(AA\mu_{2n}, AA\mu_{2n}, B\mu_{2n}, \frac{t}{3s})\}$

Since $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$ and B *and* A are compatible type (p) mapping.

Therefore, as $n \to \infty$, we get, $AA\mu_{2n} \longrightarrow B\alpha, BB\mu_{2n} \to B\alpha.$

$\leq M(\alpha, \alpha, \alpha, \frac{kt}{3s}) \star min\{M(\alpha, \alpha, B\alpha, \frac{t}{3s}), M(B\alpha, B\alpha, B\alpha, \frac{t}{3s}), M(B\alpha, B\alpha, \alpha, \frac{t}{3s})\}$

$\leq M(\alpha, \alpha, \alpha, \frac{kt}{3s}) \star min\{M(\alpha, \alpha, B\alpha, \frac{t}{3s}), M(B\alpha, B\alpha, B\alpha, \frac{t}{3s}), M(\alpha, \alpha, B\alpha, \frac{t}{3s})\}$

$\Rightarrow M(\alpha, \alpha, B\alpha, kt) \geq M(\alpha, \alpha, B\alpha, \frac{t}{3s})$

(Since, $M(\alpha, \alpha, \alpha, \frac{kt}{3s}) = 1$ and $M(B\alpha, B\alpha, B\alpha, \frac{t}{3s}) = 1$, for all $t > 0$)

Therefore, $B\alpha = \alpha$. now we will show that $A\alpha = \alpha$.

for that let $\mu = \alpha$ and $\nu = A\mu_{2n}$ then, (iv) of theorem (3.3) becomes

$M(A\alpha, A\alpha, AA\mu_{2n}, kt) \geq \min\{M(A\alpha, A\alpha, BA\mu_{2n}, t), M(AA\mu_{2n}, AA\mu_{2n}, BA\mu_{2n}, t), M(AA\mu_{2n}, AA\mu_{2n}, B\alpha, t)\}$

Since $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$, B *is continuous and* B, A are compatible type (p) such that

$AA\mu_{2n} = BB\mu_{2n} = B\alpha = \alpha$

$M(A\alpha, A\alpha, \alpha, kt) \geq \min\{M(A\alpha, A\alpha, B\alpha, t), M(\alpha, \alpha, B\alpha, t), M(\alpha, \alpha, B\alpha, t)\}$

$M(A\alpha, A\alpha, \alpha, kt) \geq \min\{M(A\alpha, A\alpha, \alpha, t), M(\alpha, \alpha, \alpha, t), M(\alpha, \alpha, \alpha, t)\},$

Since, $M(\alpha, \alpha, \alpha, t) = 1$ for all $t > 0$. Therefore, $M(A\alpha, A\alpha, \alpha, kt) \geq M(A\alpha, A\alpha, \alpha, t)$

Thus, $A\alpha = \alpha$. Hence, α is a fixed point of A and B.

Now, we prove $B\alpha = \alpha$ for N

$N(\alpha, \alpha, B\alpha, \text{kt}) \leq N(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) * N(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) * N(A\mu_{2n}, A\mu_{2n}, B\alpha, \frac{kt}{3s})$,

B is continuous and A, B are Compatible type P

such that $n \to \infty$. $AA\mu_{2n} \to B\alpha, BB\mu_{2n} \to B\alpha$,

$N(\alpha, \alpha, B\alpha, \text{kt}) \leq N(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) * N(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) * N(A\mu_{2n}, A\mu_{2n}, AA\mu_{2n}, \frac{kt}{3s})$,

$\leq N(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3s}) * max\{N(A\mu_{2n}, A\mu_{2n}, BA\mu_{2n}, \frac{t}{3s}), N(AA\mu_{2n}, AA\mu_{2n}, BA\mu_{2n}, \frac{t}{3s})$,

$N(AA\mu_{2n}, AA\mu_{2n}, B\mu_{2n}, \frac{t}{3s})\}$

Since $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$ and B *and* A are compatible type (p) mapping.

Therefore, as $n \to \infty$, we get, $AA\mu_{2n} \longrightarrow B\alpha, BB\mu_{2n} \to B\alpha$.

$\leq N(\alpha, \alpha, \alpha, \frac{kt}{3s}) * max\{N(\alpha, \alpha, B\alpha, \frac{t}{3s}), N(B\alpha, B\alpha, B\alpha, \frac{t}{3s}), N(B\alpha, B\alpha, \alpha, \frac{t}{3s})\}$

$\leq N(\alpha, \alpha, \alpha, \frac{kt}{3s}) * max\{N(\alpha, \alpha, B\alpha, \frac{t}{3s}), N(B\alpha, B\alpha, B\alpha, \frac{t}{3s}), N(\alpha, \alpha, B\alpha, \frac{t}{3s})\}$

$\Rightarrow N(\alpha, \alpha, B\alpha, \text{kt}) \leq N(\alpha, \alpha, B\alpha, \frac{t}{3s})$

(Since, $N(\alpha, \alpha, \alpha, \frac{kt}{3s}) = 0$ and $N(B\alpha, B\alpha, B\alpha, \frac{t}{3s}) = 0$, for all $t > 0$)

Therefore, $B\alpha = \alpha$. now we will show that $A\alpha = \alpha$.

for that let $\mu = \alpha$ and $\nu = A\mu_{2n}$ then, (iv) of theorem (3.3) becomes
$N(A\alpha, A\alpha, AA\mu_{2n}, \text{kt}) \leq \max\{N(A\alpha, A\alpha, BA\mu_{2n}, \text{t}), N(AA\mu_{2n}, AA\mu_{2n}, BA\mu_{2n}, \text{t}), N(AA\mu_{2n}, AA\mu_{2n}, B\alpha, \text{t})\}$
Since $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$, B is continuous *and* B, A are compatible type (p)
such that
$AA\mu_{2n} = BB\mu_{2n} = B\alpha = \alpha$
$N(A\alpha, A\alpha, \alpha, \text{kt}) \leq \max\{N(A\alpha, A\alpha, B\alpha, \text{t}), N(\alpha, \alpha, B\alpha, \text{t}), N(\alpha, \alpha, B\alpha, \text{t})\}$
$N(A\alpha, A\alpha, \alpha, \text{kt}) \leq \max\{N(A\alpha, A\alpha, \alpha, \text{t}), N(\alpha, \alpha, \alpha, \text{t}), N(\alpha, \alpha, \alpha, \text{t})\}$,
Since, $N(\alpha, \alpha, \alpha, \text{t}) = 0$ for all $t > 0$.
Therefore, $N(A\alpha, A\alpha, \alpha, \text{kt} \leq \{N(A\alpha, A\alpha, \alpha, \text{t})$
Thus, $A\alpha = \alpha$. Hence, α is a fixed point of A and B.
Uniqueness. Let α' be another coincidence and common fixed point of A and B.
then $B\alpha' = A\alpha' = \alpha'$. we get
$M(A\alpha, A\alpha, A\alpha', \text{kt}) \geq \min\{M(A\alpha, A\alpha, B\alpha', \text{t}), M(A\alpha', A\alpha', B\alpha', \text{t}), M(A\alpha', A\alpha', B\alpha, \text{t})\}$,
$M(\alpha, \alpha, \alpha', \text{kt}) \geq \min\{M(\alpha, \alpha, \alpha', \text{t}), M(\alpha', \alpha', \alpha', \text{t}), M(\alpha', \alpha', \alpha, \text{t})\}$,
(since $M(\alpha', \alpha', \alpha', \text{t})$ for all $t > 0$)
Therefore,

$M(\alpha, \alpha, \alpha', kt) \geq M(\alpha, \alpha, \alpha', t) \geq M(\alpha, \alpha, \alpha', \frac{t}{k}) \geq M(\alpha', \alpha', \alpha, \frac{t}{k^2}) \cdots \geq M(\alpha', \alpha', \alpha, \frac{t}{k^{n-1}}) \rightarrow 1$, As $n \rightarrow \infty$.

And,

$N(A\alpha, A\alpha, A\alpha', kt) \leq \max\{N(A\alpha, A\alpha, B\alpha', t), N(A\alpha', A\alpha', B\alpha', t), N(A\alpha', A\alpha', B\alpha, t)\}$,

$N(\alpha, \alpha, \alpha', kt) \leq \max\{N(\alpha, \alpha, \alpha', t), N(\alpha', \alpha', \alpha', t), N(\alpha', \alpha', \alpha, t)\}$,

(since $N(\alpha', \alpha', \alpha', t)$ for all $t > 0$)

Therefore,

$N(\alpha, \alpha, \alpha', kt) \leq N(\alpha, \alpha, \alpha', t) \leq N(\alpha, \alpha, \alpha', \frac{t}{k}) \leq N(\alpha', \alpha', \alpha, \frac{t}{k^2}) \ldots \leq N(\alpha', \alpha', \alpha, \frac{t}{k^{n-1}}) \rightarrow 0$, As $n \rightarrow \infty$.

We get $\alpha = \alpha'$. Therefor, α is the Coincidence and common fixed point of self- mappings A and B.

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