# On the asymptotically regular sequence and maps in symmetric extended Sb metric space with application

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*Abstract*—**In this paper, We introdue the concept of asymptotically regular sequence and maps with examples in symmetric extended Sb- metric spaces . As an application of asymptotically regular sequence and maps, two new results of fixed point for a contractive condition proved.**

*Index Terms*—**Fixed Point, Extended symmetric Sb metric space, asymptotically regular sequence and maps**
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## I. Introduction and Preliminaries

The concept of asymptotically regularity was introdused by Browder and Petryshyn [2] in 1966 . After that many authors see [5,6,9] done theire work for these sequence and maps and proved fixed point theorems.

On the other hand , In the histry of Mathematical analysis there are various generalization of metric spaces like as 2- Metric space Gahler [7,8], D-metric space [3,4] , G-metric Space [11,12], S-metric Space [13], b-metric Space [1], $S_b$ metric space [14] etc. In 2017, Rohen Y et al. [15] modified the definition of $S_b$ metric space, which was the real generalization of b- metric space and S-metric space and studied various properties and their applications in fixed point theory. In 2018 , Mlaki et al. [10] introduesed the notion of extended $S_b$- metric space which generalized the $S_b$-metric space.

In present paper, after introduction and preliminaris in section 2 ,we define asymptotically regular sequence and maps with example in extended symmetric $S_b$ metric space and investigate two fixed point results in this structure.

**Definition 1.1 [7,8]:** Let $X$ be a nonempty set. A generalized metric (or 2-metric) on $X$ is a function d: $X^3 \to R^+$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $d(x, y, z) \geq 0$
2. $d(x, y, z) = 0$ if and only if $x = y = z$
3. d(x, y, z) = d(p{x, y, z}), (symmetry),where p is permutation function,
4. $d(x, y, z) \leq$ d(x,y,a) + d(x,a,z) +d(a,y,z) for all $x, y, z, a \in X$.

Then the function d is called a 2- metric and the pair (X, d) is called a 2-metric space.

**Definition 1.2 [11,12]:** Let $X$ be a nonempty set. A $G$- metric on $X$ is a function $G: X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $G(x, y, z) = 0 \; if \; x = y = z$
2. $0 < G(x, y, z) \; for \; all \; x, y \in X \; with \; x \neq y,$
3. $G(x, x, y) \leq G(x, y, z) for \; all \; x, y, z \in X \; with \; x \neq y,$
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) \dots \dots ....$
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function $G$ is called an G- metric and the pair $(X, G)$ is called a $G$-metric space.

**Definition 1.3 [13]:** Let $X$ be a nonempty set. An $S$-metric on $X$ is a function $S: X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S(x, y, z) \geq 0$
2. $S(x, y, z) = 0$ if and only if $x = y = z$
3. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

Then the function $S$ is called an $S$- metric and the pair $(X, S)$ is called an $S$-metric space

**Definition 1.4** [1]: Let $X$ be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to R^+$ is a $b$-metric on $X$ if, for all $x, y, z \in X$, the following conditions hold:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, z) \leq s\, [d(x, y) + d(y, z)]$.

In this case, the pair $(X, d)$ is called a $b$-metric space

**Definition 1.5 [14]:** Let $X$ be a nonempty set and $s \geq 1$ be a given real number. An $S_b$ metric on $X$ is a function $S_b: X^3 \to [0, \infty)$, that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S_b(x, y, z) = 0$ if and only if $x = y = z$
2. $S_b(x, y, z) = S_b(x, y, z)$ for all $x, y, \in X$
3. $S_b(x, y, z) \leq s[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$

Then the function $S_b$ is called an $S_b$ metric and the pair $(X, S_b)$ is called an $S_b$ metric space.

**Remark 1.** In 2017, Rohen Y et al. [15] proved that the condition 2 of Definition 1.5 is not generally true . They have defined modification of $S_b$ metric space as follows

**Definition 1.6** [15]**:** Let $X$ be a nonempty set and $s \geq 1$ be a given real number. An $S_b$ metric on $X$ is a function $S_b: X^3 \to [0, \infty)$, that satisfies the following conditions for all $x, y, z, a \in X$.

Fig. 1. $S_b(x, y, z) = 0$ if and only if $x = y = z$

Fig. 2. $S_b(x, y, z) \leq s[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$

Then the function $S_b$ is called an $S_b$ metric and the pair $(X, S_b)$ is called an $S_b$ metric space

**Definition 1.7[10]:** Let $X$ be a nonempty set and a function $\theta: X^3 \to [1, \infty)$. An extended $S_b$ metric on $X$ is a function $S_\theta : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S_\theta(x, y, z) = 0$ if and only if $x = y = z$

2 $S_\theta(x, y, z) \leq \theta(x, y, z)\, [S_\theta(x, x, a) + S_\theta(y, y, a) + S_\theta(z, z, a)]$

Then the function $S_\theta$ is called extended $S_b$ - metric and the pair $(X, S_\theta)$ is called extended $S_b$-metric space or $S_\theta$ metric space.

**Definition 1.8[10]**. Let $(X, S_\theta)$ be a extended $S_b$-metric space then $S_\theta$ is called symmetric if

$S_\theta(x,x,y) = S_\theta(y,y,x) \quad \forall x,y \in X \; (\,Symmetric\ Property\,)$

For the definitions of convergence and Cauchy sequence in extended $S_b$ metric space reader can refer [10]

**Example 1.9[10]** Let $X = [0,\frac{1}{4}]$. Define $S_\theta : X^3 \to [0,\infty)$ by $S_\theta(x,y,z) = (\{x,y\} - z)^2$

And $\theta : X^3 \to [1,\infty)$ by $\theta(x,y,z) = \{x,y\} + z + 1$

Thus, $(X, S_\theta)$ is a completed symmetric extended $S_b$-metric space.

## II. Main results

Here we will define asymptotically regular sequences and maps in symmetric extended $S_b$-metric spaces.

**Definition 2.1.** Let $(X, S_\theta)$ be a symmetric extended $S_b$-metric space. A sequence $\{x_n\}$ in X is said to be asymptotically T- regular if

$$S_\theta(x_n, x_n, T x_n) = 0$$

OR

$$S_\theta(Tx_n, Tx_n, x_n) = 0$$

**Example 2.2.** Let $(X, S_\theta)$ be a symmetric extended $S_b$-metric space as defined in example (1.9) and T be a self map on $X$ such that $Tx = \frac{x}{3}$ and choose a sequence $\{x_n\}$, such that $x_n \to 0$ for any positive integer $n$. we deduce that

$$S_\theta(x_n, x_n, T x_n)$$

$$= \lim_{n\to\infty}(\{x_n, x_n\} - Tx_n)^2$$

$$= \lim_{n\to\infty}(x_n - \frac{x_n}{3})^2 = \lim_{n\to\infty}\frac{4\,x_n{}^2}{9} = 0 \quad As\ x_n \to 0$$

Hence $\{x_n\}$ is an asymptotically T-regular sequence in $(X, S_\theta)$

**Definaition 2.3.** Let $(X, S_\theta)$ be a symmetric extended $S_b$-metric space. A mapping of X into itself is said to be asymptotically regular at a point $x$ in X if

$$S_\theta(T^n x, T^n x, T^{n+1} x) = 0$$

OR

$$S_\theta(T^{n+1}x, T^{n+1}x, T^n x) = 0$$

**Example 2.4.** Let $(X, S_\theta)$ be a symmetric extended $S_b$-metric space as defined in example (1.9) and T be a self map on $X$ such that $Tx = \frac{x}{4}$ Where $x \in X$.then we have

$$S_\theta( T^n x, T^n x, T^{n+1}x\ ) \ = \lim_{n\to\infty}(\{T^n x, T^n x\}-T^{n+1}x)^2$$
$$= \lim_{n\to\infty}(T^n x-T^{n+1}x)^2$$
$$= \lim_{n\to\infty}((\tfrac{x}{4})^n - (\tfrac{x}{4})^{n+1})^2$$
$$= 0$$

Hence T is an asymptotically regular map at all points on X.

## III. Application

As an application of asymptotically regular sequence and maps, we present some fixed points theorems in symmetric extended $S_b$-metric space.

**Theorem 4.1:** Let $(X, S_\theta)$ be a symmetric extended $S_b$-metric space and T be a self mapping on $X$ satisfying in the inequality.

$S_\theta(Tx,Tx,Ty) \leq\ a\,S_\theta(y,y,Ty)$ $\forall x,y \in X$ and $a \geq 0$ ............................(1)

If there exists and asymptotically T-regular sequence in X ,then T has a unique fixed point.

**Proof:** let $\{x_n\}$ be an asymptotically T-regular sequences in X then by (2) of Definition 1.7 we have

$$S_\theta( x_n, x_n, x_m) \leq \theta(x_n\ ,x_n\ ,x_m\ )[\ 2\,S_\theta(x_n\ ,x_n\ ,Tx_n\ ) +\ S_\theta(x_m\ ,x_m\ ,Tx_n\ )]$$

$$\leq \theta(x_n\ ,x_n\ ,x_m\ )[\ 2\,S_\theta(x_n\ ,x_n\ ,Tx_n\ ) + \theta(x_m\ ,x_m\ ,Tx_n\ )\ [2S_\theta(x_m\ ,x_m\ ,Tx_m\ ) +\ S_\theta(Tx_n\ \ Tx_n\ ,Tx_m\ )]]$$

$$\leq \theta(x_n\ ,x_n\ ,x_m\ )\big[\ 2\,S_\theta(x_n\ ,x_n\ ,Tx_n\ ) + \theta(x_m\ ,x_m\ ,Tx_n\ )\,\{2S_\theta(x_m\ ,x_m\ ,Tx_m\ ) +\ aS_\theta(x_m\ ,x_m\ ,Tx_m\ )\}\big)\big]$$

(by condition 1 of theorem 4.1 )

Since $\{x_n\}$ is an asymptotically T-regular sequence and $m > n$.
Therefore $S_\theta(x_n\ ,x_n\ ,Tx_n\ ) \to 0$ and $S_\theta(x_m\ ,x_m\ ,Tx_m\ ) \to 0$ as $m,n \to \infty$ .
Hence $\{x_n\}$ is a Cauchy sequence.

**Existence of Fixed point:**

Consider $S_\theta(Tx,Tx,x) \leq \theta(Tx,Tx,x)[\ 2\,S_\theta(Tx,Tx,,Tx_n\ ) +\ S_\theta(x,x,Tx_n\ )]$

$$\Rightarrow\ S_\theta(Tx,Tx,x) \leq \theta(Tx,Tx,x)[\ 2a\,S_\theta(x_n\ ,x_n\ ,Tx_n\ ) +\ S_\theta(x,x,Tx_n\ )]$$

$(by\ condition\ 1\ of\ theorem\ 4.1\ )$

$$\Rightarrow S_\theta(Tx,Tx,x) \leq \theta(Tx,Tx,x)[\ 2a\,S_\theta(x_n\ ,x_n\ ,Tx_n\ ) + \theta\ (x,x,Tx_n\ )\{\ 2S_\theta(x,x,x_n\ ) +\ S_\theta(Tx_n\ ,Tx_n\ x_n\ )\ ]$$

Since $\{x_n\}$ is an asymptotically T-regular sequence therefore by definition 2.1 we have

$$S_\theta(Tx,Tx,x) \leq 0 \quad \text{as } n \to \infty$$

$$\Rightarrow Tx = x$$

**Uniqueness:**

Let $z$ be another fixed point of $T$ then

$$S_\theta(x,x,z) = S_\theta(Tx,Tx,Tz)$$

$$\leq \ aS_\theta(z,z,Tz) \text{ (by condition 1 of theorem 4.1 )}$$

$$= aS_\theta(z,z,z)$$

$$= 0$$

$$\Rightarrow x = z$$

This complestes the proof of theorem 4.1

**Theorem 4.2:** Let ($X$, $S_\theta$) be a symmetric extended $S_b$-metric space and T be a self mapping of X satisfying the condition (1) of theorem 4.1 ,

if T is asymptotically regular at some fixed point $x$ of $X$, then there exists a unique fixed point of T.

**Proof:** Let T be asymptotically regular at $x_0 \in X$. Consider the sequence $\{T^n x_0\}$ then for all $m, n \geq 1$ and $m > n$

$$S_\theta(\ T^m x_0\,,\ T^m x_0\,,\ T^n x_0) \leq\ S_\theta(\ T\, T^{m-1} x_0\,,\ TT^{m-1} x_0\,,\ TT^{n-1} x_0)$$

$$\leq\ a\, S_\theta(\ T^{n-1} x_0\,,\ T^{n-1} x_0\,,\ T^n x_0) \text{ by condition 1 of theorem 4.1}$$

Since T is asymptotically regular at some fixed point $x_0$ therefore
$S_\theta(\ T^{n-1} x_0\,,\ T^{n-1} x_0\,,\ T^n x_0)\ \to 0$
Then $S_\theta(\ T^m x_0\,,\ T^m x_0\,,\ T^n x_0) \to 0\ \ as\ \ m, n \to \infty$ .

Therefore $\{T^n x_0\}$ is a Cauchy sequence in X which is complete space .So, $\{T^n x_0\} \to x \in X$

**Existence of Fixed Point**

Now we claim that $x$ is a fixed point to T , for this we have

$$S_\theta(Tx,Tx,x) \leq \theta(Tx,Tx,x)[\ 2\ S_\theta(Tx,Tx,\ T^n x_0) +\ S_\theta(x,x,\ T^n x_0)]$$

$$= \theta(Tx,Tx,x)[\ 2S_\theta(Tx,Tx,TT^{n-1} x_0) +\ S_\theta(x,x,\ T^n x_0)]$$

$$\leq \theta(Tx,Tx,x)[\ 2a\ S_\theta(\ T^{n-1} x_0,\ T^{n-1} x_{0},,\ T^n x_0) +\ S_\theta(x,x,\ T^n x_0)]$$

by condition 1 of theorem 4.1

Since $\{T^{n-1} x_0\}$ is a subsequence of $\{T^n x_0\}\ therefore\ \ \{T^{n-1} x_0\}\ \to x$ , therefore we have

$$S_\theta(Tx,Tx,x) \leq 0 \quad \text{as } n \to \infty$$

$$\Rightarrow Tx = x$$

The uniqueness of the fixed point $x$ follows as theorem (4.1).
This completes the prrof of the Theorem (4.2).

**Remark 2** : It is worth mentioning that in theorem 4.1 and 4.2 fixed point of map are not required any condition of function $\theta$.

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