On the compatible mappings in Sb-Menger space with application

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***Abstract*—In this paper, we introduce the concept of various mappings like compatible mappings, weakly compatible mappings, semi-compatible mappings, and mappings satisfying the property (E.A.) with example. As an application of weak compatible maps and E.A property we proved a fixed-point theorem which generalize the result of Aalam et al. [2] in the new structure of Sb-Menger spaces. We have also given examples of our results.**

***Index Terms*—S-Menger Space, Sb-Menger space, Fixed Point, Compatible Maps, Weak**

***MSC*—47H10, 54H25.**

I. Introduction

Recently Sarkar et.al [9] introduce S-Menger space, very recently Keer et.al [6-8] introduce the S_b $-$Menger space which is the generalization of S and b metric space.

On the other hand, the various types of compatible maps play a very important role in fixed point theory. The notion of compatible mappings was introduced by Gerald Jungck [4] in (1986). The definition of semi-compatible mappings was defined by Y. J. Cho, B. K. Sharma and S. S. Sahu [3] in (1995). The concept of weakly compatible mappings was given by Gerald Jungck [5] in (1996) and the concept of property (E.A.) was introduced by Aamri and El Moutawakil [1] in [2002].

By inspiring above work, we investigate various types of compatible maps, weak compatible maps, semi-compatible maps and the property (E.A) in the structure of S_b $-$Menger space. Our results extend and generalize the result of Alam et.al [2] in S_b $-$Menger space.

II. Preliminaries

We recall the following definitions.

Definition 2.1.[10] A map $*: X^3 \to X$ is called continuous t-norm if it satisfies the following conditions:

1. $*(a,1,1) = a$, $*(0,0,0) = 0$;
2. $*(a,b,c) = *(a,c,b,) = *(b,c,a)$;
3. $*(a_1,b_1,c_1) \geq *(a_2,b_2,c_2)$ for $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$.

Examples of t $-$norm are

1: $x * y * z = x.y.z$ and

2: $x * y * z = min\,\{x.y.z\}$ minimum t $-$norm

Definition 2.2. (**[9]** S-Menger space)

The 3-tuple $(X, S, *)$ is said to be S-Menger space if X is a non-empty set, S is a function defined on X^3 to the set of distribution function and * is a continuous third order t-norm such that the following conditions are satisfied:

1. $S_{(x,y,z)}(0) = 0$ for all $x, y, z \in X$
2. $S_{(x,x,y)}(t) < 1$ for $t > 0$ with $x \neq y$,
3. $S_{(x,y,z)}(t) = 1$ for all $t > 0$, if and only if $x = y = z$,
4. $S_{(x,y,z)}(t) \geq * \left(S_{(x,x,a)}(t_1), S_{(y,y,a)}(t_2), S_{(z,z,a)}(t_3)\right)$,
   where $t = t_1 + t_2 + t_3$ and $t, t_1, t_2, t_3 > 0$ for all $x, y, z, a \in X$

Definition 2.3.[6] (S_b −Menger space)**:** The quadruple $(X, S_b, *, k)$ is said to be S_b-Menger space if X be a nonempty set, and $k \geq 1$ be a given real number, S_b is a distribution function on X^3 and * is a continuous third order t-norm such that the following conditions are satisfied:

1. $S_{b(x,y,z)}(0) = 0$ for all $x, y, z \in X$
2. $S_{b(x,x,y)}(t) < 1$ for $t > 0$ with $x \neq y$,
3. $S_{b(x,y,z)}(t) = 1$ for all $t > 0$, if and only if $x = y = z$,
4. $S_{b(x,y,z)}(kt) \geq [* \left(S_{b(x,x,a)}(t_1), S_{b(y,y,a)}(t_2), S_{b(z,z,a)}(t_3)\right)]$,
   where $t = t_1 + t_2 + t_3$ and $t, t_1, t_2, t_3 > 0$ for all $x, y, z, a \in X$

Definition 2.4. [6] The $S_b -$ Menger space is called symmetric if $S_{b(x,x,y)}(t) = S_{b(y,y,x)}(t)$

Definition 2.5. [6] Let $(X, S_b, *, k)$ be a symmetric $S_b -$ Menger space then a sequence $\{x_n\} \in X$ is said to be convergent to a point $x \in X$ if $S_{b(x_n,x_n,x)}(t) = 1$ for all $t > 0$.

Definition 2.6. [6] Let $(X, S_b, *, k)$ be a symmetric $S_b -$ Menger space then a sequence $\{x_n\} \in X$ is called Cauchy sequence if $x \in X$ if $S_{b(x_n,x_n,x_{n+s})}(t) = 1$ for all $t, s > 0$.

Lemma 2.7.[6] Let $(X, S_b, *, k)$ be an S_b-Menger space with continuous third order t-norm. Then $S_{b(x,x,y)}(t)$ is non-decreasing with respect to t, for all $x, y \in X$
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III. Main Results

Definition 3.1. Let S and T maps from an S_b −Menger space $(X, S_b, *, k)$ into itself. The maps S and T are said to be compatible, if for all $t > 0$.

$$S_{b(STx_n,STx_n,TSx_n)}(t) = 1$$

whenever $\{x_n\}$ is a sequence in X such that $Sx_n = Tx_n = z$ for some $z \in X$.

Example 3.2. Let $X = [2,20)$. For each $t \in (0, \infty)$ and for all $x, y, z \in X$, define

$S_{b(x,y,z)}(t) = \{0,\ t = 0\ \frac{t}{t+[|x-z|+|y-z|]^2},\ t \geq 0$

Clearly $(X, S_b, *, k)$ is an S_b-Menger space, where $*$ is defined by $a * b * c = abc$.

Let g and f be self-maps of X defined as

$S(x) = \{2,\ if\ x = 2\ or\ x > 5\ 6,\ if\ 2 < x \leq 5$ $T(x) = \{2\ \ if\ x = 2\ or\ x > 5\ 12\ \ if\ 2 < x \leq 5\ \frac{x+1}{3}\ \ if\ x > 5.$

Let sequence $\{x_n\}$ be defined as x_n = 5+ 1/n , n ≥ 1 then we have

$Sx_n = \lim_{n\to\infty} Tx_n$=2.

Hence, S and T satisfy the property (E.A.). Also, $\lim_{n\to\infty} S_{b(STx_n, STx_n, TSx_n)}(t) = \frac{t}{t+[|2-2|+|2-2|]^2} = \frac{t}{t+0} = 1$

This shows that T and f are compatible.

Definition 3.3. Let S and T be maps from an S_b −Menger space space *(X, S_b, *,k)* into itself. The maps are said to be weakly compatible, if they commute at their coincidence points, that is, *Sz = T z* implies that *ST z = T Sz.*

Definition 3.4. Let S and T maps from an S_b-Menger space *(X, S_b, *,k)* into itself. The maps S and T are said to be semi-compatible, if for all $t > 0$.

$$S_{b(STx_n, STx_n, TSz)}(t) = 1$$

whenever $\{x_n\}$ is a sequence in X such that $gx_n = fx_n = z$ for some $z \in X$.

Note that the semi-compatibility of the pair *(S, T)*, need not imply the semi-compatibility of *(T, S)*.

The following is an example of a pair of self-maps *(S, T)* which is compatible but not semi-compatible. Further, it is also seen that the semi-compatibility of the pair *(S, T)* need not imply the semi-compatibility of *(T, S)*.

Example 3.5. Let $X = [0,1]$ and let *(X, S_b, *,k)* be the S_b-Menger space with

$S_{b(x,y,z)}(t) = \{e^{[\frac{[|x-z|+|y-z|]^2}{t}]^{-1}},\ t > 0\ 0\ \ \ t = 0$ for all $x, y, z \in X, t > 0$.

Example 3.6. Define self-maps S as follow:

$$S(x) = \{x,\quad when\ 0 \leq x < \frac{1}{2}\ 1,\quad when\quad x \geq \frac{1}{2}$$

Let I be the identity map on X and $x_n = \frac{1}{2} - \frac{1}{n}$. Then, $\{Ix_n\} = \{x_n\} \to \frac{1}{2}$ and $\{Sx_n\} = \{x_n\} \to \frac{1}{2}$. Thus, $\{ISx_n\} = \{Sx_n\} = \frac{1}{2} \neq S\{\frac{1}{2}\}$.

Hence (*IS*) is not semi-compatible.

Again as (*I, S*) is commuting, it is compatible. Further, for any sequence $\{x_n\}$ in X such that $\{x_n\} \to x$ and $\{Sx_n\} \to x$, we have $\{SIx_n\} = \{Sx_n\} \to x = I\,x$. Hence (*SI*) is always semi-compatible

Example 3.7. Let $X = [0,2]$ and *(X, S_b, *)* be an S_b-Menger space, where the definition of * and S_b are same as defined in example 3.3. Define self-maps A and G on X as follows.

$A(x) = \{2,\ if\ 0 \leq\ x \leq 1\ \frac{x}{2},\ if\ 1 < x \leq 2 \quad G(x) = \{2 \quad ,\ if\ \ x = 1\ \frac{x+3}{5} \quad ,\ otherwise$

and $x_n = 2 - \frac{1}{2n}, n \geq 1$. Then we have $G(1) = A(1) = 2$ and $G(2) = A(2) = 1$. Also $GA(1) = AG(1) = 1$ and $GA(2) = AG(2) = 2$. Thus (A, G) is weak compatible. Again $Ax_n = 1 - \frac{1}{4n}, Gx_n = 1 - \frac{1}{10n}$.

Thus $Ax_n \to 1, Gx_n \to 1$.

Hence $u = 1$.

Further, $GAx_n = \frac{4}{5} - \frac{1}{20n}, AGx_n = 2$.

Now $S_{b(AGx_n,AGx_n,Gu)}(t) = S_{b(2,2,2)}(t) = 1$

$$S_{b(AGx_n,AGx_n,GAx_n)}(t) = S_{b\left(2,2,\frac{4}{5}-\frac{1}{20n}\right)}(t)$$

$$= \frac{t}{t + [\left|2 - \frac{4}{5}\right| + \left|2 - \frac{4}{5}\right|]^2}$$

$$= \frac{t}{t + [\frac{12}{5}]^2} < 1 \qquad \forall\, t > 0.$$

Hence (A, G) is semi-compatible but it is not compatible.

Remark 3.8. The above example gives an important aspect of semi-compatibility as the pair of self-maps (*IS*) is commuting; hence it is weakly commuting, compatible and weak compatible yet it is not semi-compatible. Further, it is to be noted that the pair (*S, I*) is semi-compatible but (*I, S*) is not semi-compatible here. The following is an example of a pair of self-maps (A, S) which is semi-compatible but not compatible.

Definition 3.9. Let *S* and *T* be two self-maps of an S_b −Menger space *(X, S_b *,k)*. We say that *S* and *T* satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ such that $Sx_n = Tx_n = = z$ for some *z* ∈*X*.

Remark 3.10. Note that the weakly compatible and property (E.A.) are independent to each other. See the following example

Example 3.11. Let *(X, S_b, *,k)* be an S_b-Menger space where $X = [0,1]$ and S_b is defined as in 3.3. Defined $g, f: X \to X$ by

$g(x) = 1 - x$, if $x \in [0, \frac{1}{2}]$ and $g(x) = 0, if\ \ x \in (\frac{1}{2}, 1]$

$f(x) = \frac{1}{2}$, if $x \in [0, \frac{1}{2}]$ and $f(x) = \frac{3}{4}, if\ x \in (\frac{1}{2}, 1]$

Then, for the sequence $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}, n \geq 2$. we have

$g\left(\frac{1}{2} - \frac{1}{n}\right) = \left(\frac{1}{2} + \frac{1}{n}\right) \ = \frac{1}{2} = f\left(\frac{1}{2} - \frac{1}{n}\right)$

Thus, the pair (g, f) satisfies property *(E.A)*. Further, g and f are weakly compatible since $x = \frac{1}{2}$ is their unique coincidence point and $gf\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = fg\left(\frac{1}{2}\right)$.We further observe that

$$S_b(gf\left(\frac{1}{2}-\frac{1}{n}\right), gf\left(\frac{1}{2}-\frac{1}{n}\right), fg\left(\frac{1}{2}-\frac{1}{n}\right)) = \frac{t}{t + [2\left|gf\left(\frac{1}{2}-\frac{1}{n}\right) - fg\left(\frac{1}{2}-\frac{1}{n}\right)\right|]^2}$$

$$= \frac{t}{t + [2\left|\frac{1}{2}-\frac{3}{4}\right|]^2} = \frac{t}{t + [\frac{1}{2}]^2} \neq 1$$

showing that the pair (g, f) is noncompatible.

A Class of Implicit Relation: Let ϕ be the set of all real continuous functions $\phi : [R^+]^4 \to R$, nondecreasing in first argument and satisfying the following conditions.

1. For $u, v \geq 0, \phi(u, v, v, u) \geq 0 \; or \; \phi(u, v, u, v) \geq 0 \; implies \; u \geq v$ …. (a)
2. $\phi(u, v, 1,1) \geq 0 \; implies \; that \; u \geq 1$ ………….. (b)

Application of compatible maps and (E.A) property: As an application of weak compatible maps and the property (EA), we prove the fixed-point theorem of Alam et.al [2] in S_b-Menger space.

Theorem 3.12: Let $L, M, N \; and \; O$ be self-maps of an S_b −Menger space *(X, S_b, ∗,k)* satisfying the following conditions:

$L(X) \subseteq O(X), M(X) \subseteq N(X)$; ………… (3.12.1)

$(L, N) and \; (M, O)$ are weakly compatible pairs ………………. (3.12.2)

$(L, N) \; or \; (M, O)$ satisfies the property (E.A) ; ………………….(3.12.3)

For some $\phi \in \emptyset$, there exist $k \in (0,1)$ such that for all $x, y \in X, t > 0$

$\phi(S_{b(Lx,Lx,My)}(kt), S_{b(Nx,Nx,Oy)}(t), S_{b(Lx,Lx,Nx)}(t), S_{b(Ly,Ly,Oy)}(t)) \geq 0$….. (3.12.4)

If the range of one of the maps *L, M, N* or *O* is a complete subspace of *X*, then *L, M, N* and *O* have a unique common fixed point in *X.*

Proof: I f the pair (M, O) satisfies the property (E.A) then there exist a sequence $\{x_n\}$ such that $Mx_n \to z$ and $Ox_n \to z$ for some $z \in X$ as $n \to \infty$.

Since $M(X) \subseteq N(X)$,there exist in X a sequence $\{y_n\}$such that $Mx_n \subseteq Nx_n$.

Hence $Ny_n \to z$ as $n \to \infty$.

Now we claim that $Ly_n \to z$ as $n \to \infty$.Suppose $Ly_n \to w (\neq z) \in X$, then by (3.12.4),

we have

$$\phi(S_{b(Ly_n,Ly_n,Mx_n)}(kt), S_{b(Ny_n,Ny_n,Ox_n)}(t), S_{b(Ly_n,Ly_n,Ny_n)}(t), S_{b(Mx_n,Mx_n,Ox_n)}(t)) \geq 0$$

that is,

$$\phi(S_{b(Ly_n,Ly_n,Mx_n)}(kt), S_{b(Mx_n,Mx_n,Ox_n)}(t), S_{b(Ly_n,Ly_n,Mx_n)}(t), S_{b(Mx_n,Mx_n,Ox_n)}(t)) \geq 0$$

As ϕ is nondecreasing in the first argument, we have

$$\phi(S_{b(Ly_n,Ly_n,Mx_n)}(t), S_{b(Mx_n,Mx_n,Ox_n)}(t), S_{b(Ly_n,Ly_n,Mx_n)}(t), S_{b(Mx_n,Mx_n,Ox_n)}(t)) \geq 0$$

Using (A), we get $S_{b(Ly_n,Ly_n,Mx_n)}(t) \geq S_{b(Mx_n,Mx_n,Ox_n)}(t)$.

Letting $n \to \infty$,$S_{b(w,w,z)}(t) \geq 1$ for all $t > 0$.

Hence $S_{b(w,w,z)}(t) = 1$

Thus $w = z$.

This shows that $Ly_n \to z$ as $n \to \infty$.

suppose that $N(X)$ is complete subspace of X.

Then $z = Nu$ for some $u \in X$.

Subsequently, we have $Ly_n \to Nu$, $Mx_n \to Nu$, $Ox_n \to Nu$ and $Ny_n \to Nu$ as $n \to \infty$.

By (3.12.4), we have

$$\phi(S_{b(Lu,Lu,Mx_n)}(kt), S_{b(Nu,Nu,Ox_n)}(t), S_{b(Lu,Lu,Nu)}(t), S_{b(Mx_n,Mx_n,Ox_n)}(t)) \geq 0$$

Letting $n \to \infty$

$$\phi(S_{b(Lu,Lu,Nu)}(kt), 1, S_{b(Lu,Lu,Nu)}(t), 1) \geq 0$$

As ϕ is nondecreasing in the first argument, we have

$$\phi(S_{b(Lu,Lu,Nu)}(t), 1, S_{b(Lu,Lu,Nu)}(t), 1) \geq 0$$

By using (A), we get $S_{b(Lu,Lu,Nu)}(t) \geq 1$ for all $t > 0$.

Hence, $S_{b(Lu,Lu,Nu)}(t) = 1$.

Thus, $Lu = Nu$.

The weak compatibility of L and N implies that $LNu = NLu$ then $LLu = LNu = NLu = NNu$.

On the other hand, since $L(X) \subseteq O(X)$, there exists a $v \in X$ such that $Lu = Ov$.

we show that $Ov = Mv$.

By (3.12.4), we have

$$\phi(S_{b(Lu,Lu,Mv)}(kt), S_{b(Nu,Nu,Ov)}(t), S_{b(Lu,Lu,Nu)}(t), S_{b(Mv,Mv,Ov)}(t)) \geq 0$$

that is ,

$$\phi(S_{b(Ov,Ov,Mv)}(kt), 1,1, S_{b(Mv,Mv,Ov)}(t)) \geq 0$$

As ϕ is nondecreasing in the first argument, we have

$\phi(S_{b(Ov,Ov,Mv)}(kt), 1,1, S_{b(Mv,Mv,Ov)}(t)) \geq 0$ **[As** $S_{b(x,x,y)}(t) = S_{b(y,y,x)}(t)$]

Using (a), we get $S_{b(Ov,Ov,Mv)}(t) \geq 1$ for all $t > 0$.

Hence,

$S_{b(Ov,Ov,Mv)}(t) = 1$.

Thus, $Mv = Ov$.

This implies $Lu = Nu = Ov = Mv$. The weak compatibility of $M\ and\ O$ implies that $MOv = OMv$ and then $OOv = OMv = MOv = MMv$.

Now, we will show that Lu is a common fixed point of $L, M, N\ and\ O$.

In view of (3.12.4) it follows

$$\phi(S_{b(LLu,LLu,Mv)}(kt), S_{b(NLu,SEu,Tv)}(t), S_{b(LLu,LLu,NLu)}(t), S_{b(Mv,Mv,Ov)}(t)) \geq 0$$

that is,

$$\phi(S_{b(LLu,LLu,Lu)}(kt), S_{b(LLu,LLu,Lu)}(t), 1,1) \geq 0$$

As ϕ is nondecreasing in the first argument, we have

$$\phi(S_{b(LLu,LLu,Lu)}(t), S_{b(LLu,LLu,Lu)}(t), 1,1) \geq 0$$

Using (B), we get $S_{b(LLu,LLu,Lu)}(t) \geq 1$ for all $t > 0$. Hence $S_{b(LLu,LLu,Lu)}(t) = 1$

Thus $LLu = Lu$.

Therefore, $Lu = LLu = NLu$ and Lu is a common fixed point of L and N. Similarly, we prove that Mv is a common fixed point of M and O. Since $Lu = Mv$, we conclude that Lu is a common fixed point of $L, M, N\ and\ O$. The proof is similar when $O(X)$ is assumed to be a complete subspace of X. The cases in which $L(X)$ or $M(X)$ is a complete subspace of X are similar to the cases in which $O(X)$ or $N(X)$ respectively, is complete since $L(X) \subseteq O(X)$, $M(X) \subseteq N(X)$.

$Lu = Mu = Ou = Nu = u$ and $Lv = Mv = Nv = Ov = v$ then (3.12.4) gives

$$\phi(S_{b(Lu,Lu,Mv)}(kt), S_{b(Nu,Nu,Ov)}(t), S_{b(Lu,Lu,Nu)}(t), S_{b(Mv,Mv,Ov)}(t)) \geq 0$$

that is ,

$$\phi(S_{b(u,u,v)}(kt), S_{b(u,u,v)}(t), 1,1) \geq 0$$

sing (b), we get $S_{b(u,u,v)}(t) \geq$ for all t > 0.

Hence, $S_{b(u,u,v)}(t) = 1$.

Thus, $u = v$.

Therefore, the common fixed point is unique.

The following example illustrates our result.

Example 3.13. Let *(X,* S_b*, ∗,k)* be an S_b-Menger space as defined in example (3.2)

Defined $L, M, N, O: X \to X$ by

$$L(x) = \{2, \qquad when\ x = 2\ 1, \qquad when\ 2 < x < 20$$

$$M(x) = \{2, \qquad when\ x = 2\ 7, \qquad when\ 2 < x < 20$$

$N(x) = \{2 \quad when\ x = 2\ 6 \quad when\ 2 < x \leq 10\ (x - 7) \quad when\ 10 < x < 20$

$O(x) = \{2 \quad when\ x = 2\ 3 \quad when\ 2 < x \leq 10\ (x - 3) \quad when\ 10 < x < 20$

Then *L,M,N* and *O* satisfy all the conditions of theorem with $k \in (0, 1)$ and have a unique common fixed-point $x = 2$. Clearly *(L,N)* and *(M,O)* are weakly compatible since they commute at their coincidence points. Let sequence $\{x_n\}$ be defined as $x_n = 10 + \frac{1}{n}, n \geq 1$, then we have $M\ x_n$=$M\ x_n$=7. Hence, *M* and *O* satisfy the property *(E.A.)*.

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