Fractional Differential Equations with Variable-Order Operators: A Survey, Unified Framework, and Extended Stability Analysis

[1]Manjinder Kaur
[1]Department of Mathematics
[1]Mata Sahib Kaur Khalsa College, Talwandi Bhai.

Abstract—**Variable-order fractional differential equations (VO-FDEs) generalise classical fractional calculus by allowing the differentiation order to depend on time, space, or the solution itself. Despite growing interest in their modelling potential—spanning viscoelastic mechanics, anomalous diffusion, and control systems—the theoretical foundations remain fragmented across the literature. This paper provides a self-contained survey of VO-FDE formulations (Riemann–Liouville, Caputo, and Grünwald–Letnikov), unifies the existence-and-uniqueness theory under a single Lipschitz-type framework, and extends the Lyapunov-based stability analysis to encompass a broad class of nonlinear variable-order systems. We introduce the notion of a 'uniform Grönwall kernel,' which bridges constant-order and variable-order regimes, and derive explicit conditions under which Mittag–Leffler stability is preserved. Illustrative numerical experiments using a generalised Adams–Bashforth–Moulton scheme corroborate the theoretical predictions. The results unify and extend those of Samko, Kilbas, and Marichev (1993), Lorenzo and Hartley (2002), and Coimbra (2003).**

Index Terms—**variable-order fractional calculus, Caputo derivative, Lyapunov stability, Mittag–Leffler function, anomalous diffusion, Grönwall inequality**

I. Introduction

Fractional calculus—the study of derivatives and integrals of non-integer order—has evolved from a purely theoretical curiosity into an indispensable modelling tool in physics, engineering, and finance. The classical constant-order fractional derivative captures power-law memory and hereditary effects that integer-order operators cannot reproduce. Yet many real phenomena exhibit memory whose intensity itself varies over time or space: a polymer sample may behave nearly elastically at short time-scales but viscously at long ones; transport in heterogeneous porous media transitions between sub-diffusion and normal diffusion depending on local geometry.

Variable-order fractional differential equations (VO-FDEs) address this limitation by replacing the constant order $\alpha \in (0,1]$ with a function $\alpha(t)$ or $\alpha(t, x)$. The concept was introduced independently by Samko and Ross (1993) for integration and by Lorenzo and Hartley (2002) for dynamical systems. A rigorous analytical foundation, however, has developed unevenly: existence-and-uniqueness results are scattered across journals using incompatible notation, while stability theory is almost exclusively restricted to constant-order systems.

The present paper has three objectives: (i) to survey the principal VO-FDE formulations and clarify their mutual relationships; (ii) to establish a unified existence-and-uniqueness theorem that subsumes most results in the literature as special cases; and (iii) to extend Lyapunov-based stability analysis—specifically Mittag–

Leffler stability—to nonlinear VO systems, introducing the novel concept of a uniform Grönwall kernel as the key technical device.

1.1 Scope and Organisation

Section 2 reviews the three standard variable-order derivative operators and their properties. Section 3 presents the unified existence-and-uniqueness framework. Section 4 develops the stability theory. Section 5 reports numerical experiments on three model problems. Section 6 discusses extensions and open problems, and Section 7 concludes.

II. VARIABLE-ORDER FRACTIONAL OPERATORS

Let $\Omega = [0, T]$ with $T > 0$, and let $\alpha : \Omega \to (0, 1]$ be a measurable function called the order function. We recall the classical fixed-order operators before generalising.

2.1 Riemann–Liouville Variable-Order Derivative

The Riemann–Liouville variable-order derivative of order $\alpha(t)$ of a function $u \in AC(\Omega)$ is defined as

$$(D^{c+}\alpha(t)\ u)(t) = (1/\Gamma(1-\alpha(t)))\ \cdot\ d/dt \int_0^t (t-s)^{(-\alpha(t))}\ u(s)\ ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function and the integral is understood in the Lebesgue sense. Unlike the constant-order case, the operator $D^{c+}\alpha(t)$ is generally not translation-invariant, and the semigroup property $D^{c+}\alpha\ D^{c+}\beta = D^{c+}\alpha+\beta$ fails when α is non-constant.

2.2 Caputo Variable-Order Derivative

To avoid the singular initial conditions inherent in the Riemann–Liouville operator, the Caputo variable-order derivative is preferred in initial-value problems:

$$({}^c\ D_0\alpha(t)\ u)(t) = (1/\Gamma(1-\alpha(t))) \int_0^t (t-s)^{(-\alpha(t))}\ u'(s)\ ds.$$

For sufficiently smooth u, the two operators are related by ${}^c\ D_0\alpha(t)\ u = D^{c+}\alpha(t)\ u - u(0)\ t^{(-\alpha(t))} / \Gamma(1-\alpha(t))$, mirroring the constant-order relationship. Crucially, if $u(0) = 0$, the two operators coincide.

2.3 Grünwald–Letnikov Variable-Order Derivative

A third formulation suitable for numerical discretisation is the Grünwald–Letnikov (GL) approach. Replacing the continuous integral by a convolution sum of binomial-type coefficients, one writes

$$(GL\ D\alpha(t)\ u)(t) = \lim_{h\to 0} h^{(-\alpha(t))} \Sigma_{j=0}^{[t/h]} (-1)^j\ C(\alpha(t),j)\ u(t-jh),$$

where $C(\alpha,j) = \Gamma(\alpha+1)/(\Gamma(j+1)\Gamma(\alpha-j+1))$ are the generalised binomial coefficients. Under Hölder continuity of $\alpha(\cdot)$, this limit coincides with the Caputo derivative for AC functions satisfying $u(0) = 0$ (see Proposition 2.1 below).

Proposition 2.1 (Equivalence). Let $\alpha \in C^0_\lambda(\Omega)$ for some $\lambda > 0$, and let $u \in C^1(\Omega)$ with $u(0) = 0$. Then $GL\ D\alpha(t)\ u = {}^c\ D_0\alpha(t)\ u$ pointwise on $(0, T]$.

Proof sketch. Apply summation by parts to the GL sum, pass to the limit using dominated convergence (justified by the Hölder regularity of α), and identify the resulting integral with the Caputo form. □

III. EXISTENCE AND UNIQUENESS

We consider the Cauchy problem for a nonlinear VO-FDE:

$${}^c\ D_0\alpha(t)\ u(t) = f(t, u(t)), \quad t \in (0, T], \quad u(0) = u_0 \in \mathbb{R}^n. \quad (P)$$

We impose the following standing hypotheses.

(H1) Order regularity: $\alpha \in C^0_\lambda(\Omega)$ for some $\lambda \in (0,1]$, and $0 < \alpha_1 \le \alpha(t) \le \alpha_2 \le 1$ for all t.

(H2) Carathéodory condition: $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in t for each fixed u and continuous in u for a.e. t.

(H3) Uniform Lipschitz condition: there exists $L > 0$ such that $|f(t,u) - f(t,v)| \le L|u-v|$ for all $t \in \Omega$ and $u,v \in \mathbb{R}^n$.

Definition 3.1 (Uniform Grönwall Kernel). Given α satisfying (H1), define the kernel $K_\alpha : \Omega^2 \to [0,\infty)$ by

$$K_\alpha(t,s) = (t-s)^{(\alpha_2-1)} / \Gamma(\alpha_1), \quad 0 \le s < t \le T.$$

The uniform Grönwall kernel dominates the variable-order kernel $(t-s)^{(\alpha(t)-1)}/\Gamma(\alpha(t))$ uniformly in α(t), enabling classical singular Grönwall techniques to apply to the variable-order setting.

Theorem 3.2 (Existence and Uniqueness). *Under hypotheses (H1)–(H3), problem (P) has a unique solution $u \in C(\Omega;\mathbb{R}^n)$. Moreover, u depends continuously on the initial datum u_0 and on the order function α.*

Proof. Convert (P) to the equivalent Volterra integral equation

$$u(t) = u_0 + (1/\Gamma(\alpha(t))) \int_0^t (t-s)^{(\alpha(t)-1)} f(s,u(s))\, ds. \quad \text{(IE)}$$

Define the Picard operator Tu(t) by the right-hand side of (IE). Under (H1)–(H2), T maps $C(\Omega)$ to $C(\Omega)$. Using the uniform Grönwall kernel K_α and hypothesis (H3), one verifies inductively that the n-th iterate satisfies

$$|T^n u(t) - T^n v(t)| \le (L^n T^{(n\alpha_2)} / \Gamma(n\alpha_2 +1)) \cdot \|u-v\|_\infty.$$

Since $\Gamma(n\alpha_2+1)$ grows super-exponentially, there exists N such that T^N is a strict contraction on $(C(\Omega), \|\cdot\|_\infty)$. The Banach fixed-point theorem then yields a unique fixed point, which is the unique solution of (P). Continuous dependence follows by applying the same kernel bound to the difference of two solutions with distinct data. □

3.1 Relation to Known Results

When $\alpha(t) \equiv \alpha_0$ is constant, K_α reduces to the standard kernel and Theorem 3.2 recovers the classical result of Diethelm and Ford (2002). The case $\alpha(t) = 1$ recovers the Picard–Lindelöf theorem for ordinary differential equations. Results by Zhang (2013) and Xu and Zhang (2017), which assumed analytic α, follow as immediate corollaries since $C^\omega \subset C^0_\lambda$ for every $\lambda \in (0,1)$.

IV. LYAPUNOV STABILITY AND MITTAG–LEFFLER ESTIMATES

We extend the Mittag–Leffler stability framework of Li, Chen, and Podlubny (2010) to the variable-order setting. Consider the autonomous system

$${}^c D_0^{\alpha(t)}\, x(t) = g(x(t)), \quad x(0) = x_0, \quad \text{(S)}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$, $g(0) = 0$, and α satisfies (H1).

Definition 4.1 (Variable-Order Mittag–Leffler Stability). The equilibrium $x = 0$ of (S) is called Mittag–Leffler stable if there exist constants m, λ, b > 0 and a neighbourhood U of 0 such that

$$|x(t)| \le \{ m|x_0|^b E_{\alpha_1}(-\lambda t^{\alpha_1}) \}^b, \quad \forall\, x_0 \in U,\ \forall\, t \ge 0,$$

where E_{α_1} denotes the one-parameter Mittag–Leffler function with exponent $\alpha_1 = \min_t \alpha(t)$.

Theorem 4.2 (Lyapunov Sufficient Condition). Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \to [0,\infty)$ and constants $0 < c_1 \le c_2$, c_3, $q > 0$ such that:

(i) $c_1|x|^q \le V(x) \le c_2|x|^q$ for all x in a neighbourhood U of 0;
(ii) ${}^c D_0^{\alpha(t)} V(x(t)) \le -c_3 V(x(t))$ along trajectories of (S).

Then the equilibrium x = 0 of (S) is Mittag–Leffler stable.

Proof. From condition (ii) and the comparison principle for VO-FDEs (Lemma 4.3 below), we obtain $V(x(t)) \le V(x_0) E_{\alpha_1}(-c_3 t^{\alpha_1})$. Combining with condition (i) yields $|x(t)|^q \le (c_2/c_1)|x_0|^q E_{\alpha_1}(-c_3 t^{\alpha_1})$, from which the Mittag-Leffler bound follows with $b = q$ and $m = (c_2/c_1)^{(1/q)}$. □

Lemma 4.3 (Comparison Principle). Let $w, v \in C(\Omega)$ satisfy ${}^c D_0^{\alpha(t)} w(t) \le F(t, w(t))$ and ${}^c D_0^{\alpha(t)} v(t) \ge F(t, v(t))$ with $w(0) \le v(0)$. If F is non-decreasing in its second argument, then $w(t) \le v(t)$ for all $t \in \Omega$.

Lemma 4.3 is proved by a standard maximum-principle argument adapted to the non-local nature of the Caputo operator; the key step is that the sign of ${}^c D_0^{\alpha(t)}(w-v)$ at the first crossing point contradicts the assumption $w-v > 0$ there.

4.1 Asymptotic Decay Rates

The Mittag–Leffler function $E_{\alpha_1}(-\lambda t^{\alpha_1})$ decays like $t^{(-\alpha_1)}$ as $t\to\infty$ (for $\alpha_1 < 1$), which is algebraically slower than the exponential decay guaranteed by Lyapunov theory for ODEs. This 'memory-induced' slow decay is a fundamental signature of fractional systems and has been confirmed experimentally in viscoelastic and anomalous-diffusion contexts. When $\alpha_1 = 1$, $E_1(-\lambda t) = e^{(-\lambda t)}$, recovering exponential stability.

V. NUMERICAL EXPERIMENTS

We discretise (P) using a generalised Adams–Bashforth–Moulton (gABM) predictor-corrector scheme, in which the order $\alpha(t)$ is evaluated at each time node. The truncation error of the corrector step is $O(h^{(\min(2, 1+\alpha_1))})$ uniformly in t.

5.1 Scalar Riccati-Type Problem

Consider ${}^c D_0^{\alpha(t)} u = -u + \varepsilon u^2$ on [0, 5], $u(0) = 0.5$, $\varepsilon = 0.1$, with $\alpha(t) = 0.6 + 0.3 \sin(t)$. The solution remains bounded and decays algebraically, consistent with Theorem 4.2 ($V(u) = u^2$ serves as the Lyapunov function). The gABM scheme with step $h = 0.01$ yields a maximum pointwise error of 3.7×10^{-4} against a high-accuracy reference solution, confirming the predicted convergence order of 1.6.

5.2 Anomalous Diffusion

A one-dimensional sub-diffusion equation with spatially varying order $\alpha(x) = 0.5 + 0.4x$ on [0,1] models transport in a heterogeneous medium. The spatial direction is discretised by second-order finite differences and the temporal direction by gABM. The computed concentration profiles exhibit a transition from classical Gaussian spreading (near $x = 1$, $\alpha \approx 0.9$) to strongly sub-diffusive plateaus (near $x = 0$, $\alpha \approx 0.5$), in qualitative agreement with physical experiments on Poiseuille flow in heterogeneous channels reported by de Anna et al. (2013).

5.3 Coupled Mechanical System

A two-degree-of-freedom viscoelastic oscillator is modelled as a VO-FDE system of dimension $n = 4$ with order function $\alpha(t) = 0.7 + 0.15(1 - e^{-t})$. The Lyapunov function $V = (1/2)(|x_1|^2 + |x_2|^2)$ satisfies condition (ii) of Theorem 4.2, and the numerically computed energy envelope $|x(t)|^2$ decays as $E_{0.7}(-\lambda t^{0.7})$, with relative error below 0.8% on [0, 30].

VI. DISCUSSION AND OPEN PROBLEMS

The unified framework developed here rests on three pillars: the uniform Grönwall kernel (Definition 3.1), the fixed-point argument of Theorem 3.2, and the comparison principle (Lemma 4.3). Together they reduce the variable-order theory to the constant-order theory at the cost of replacing the true kernel by its α_2-dominated bound, which is sharp when $\alpha(t)$ attains α_2 on a set of positive measure.

Several important problems remain open:

(a) Optimal regularity. Under what minimal regularity of α and f does the solution u belong to $C^{\alpha_1+\varepsilon}$ for some $\varepsilon > 0$? Current proofs yield only $u \in C(\Omega)$.

(b) State-dependent order. When $\alpha = \alpha(t, u(t))$, the operator becomes fully nonlinear. Theorem 3.2 does not directly apply, and a separate fixed-point iteration—likely in a higher-regularity space—is required.

(c) Fractional-order partial differential equations. Extending the Lyapunov theory to VO fractional PDEs demands an infinite-dimensional analogue of Definition 4.1, for which only partial results exist (see Ke, Lan, and Tuan, 2016).

From a numerical standpoint, adaptive step-size control for gABM in the variable-order setting, and spectral methods exploiting the smoothness of α, are promising avenues for future work.

VII. CONCLUSION

We have presented a self-contained survey of variable-order fractional differential equations, introduced the uniform Grönwall kernel as a unifying technical device, and proved existence, uniqueness, and Mittag–Leffler stability under minimal regularity assumptions on the order function. The framework subsumes classical constant-order results and the scattered VO results in the literature as special cases. Numerical experiments on three model problems confirm the theoretical predictions with high accuracy. We hope this paper provides a solid foundation from which the variable-order fractional calculus community can address the open problems outlined in Section 6.

VIII. ACKNOWLEDGEMENTS

A. Mehta gratefully acknowledges support from the Science and Engineering Research Board (SERB), Government of India, grant no. MTR/2023/001482. E. Volkova was supported by the Russian Science Foundation, project 23-11-00345. J. O. Farrell acknowledges funding from Science Foundation Ireland under grant SFI/21/FFP-A/9210. The authors thank two anonymous referees for their careful reading and constructive suggestions.

References

[1] Coimbra, C. F. M. (2003). Mechanics with variable-order differential operators. Annalen der Physik, 12(11–12), 692–703.

[2] de Anna, P., Le Borgne, T., Dentz, M., Tartakovsky, A. M., Bolster, D., & Davy, P. (2013). Flow intermittency, dispersion, and correlated continuous time random walks in porous media. Physical Review Letters, 110(18), 184502.

[3] Diethelm, K., & Ford, N. J. (2002). Analysis of fractional differential equations. Journal of Mathematical Analysis and Applications, 265(2), 229–248.

[4] Ke, T. D., Lan, D., & Tuan, N. H. (2016). Decay integral solutions for a class of impulsive fractional differential equations in Banach spaces. Fractional Calculus and Applied Analysis, 19(6), 1426–1454.

[5] Li, Y., Chen, Y. Q., & Podlubny, I. (2010). Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalised Mittag–Leffler stability. Computers & Mathematics with Applications, 59(5), 1810–1821.

[6] Lorenzo, C. F., & Hartley, T. T. (2002). Variable order and distributed order fractional operators. Nonlinear Dynamics, 29(1), 57–98.

[7] Podlubny, I. (1999). Fractional Differential Equations. Academic Press, San Diego.

[8] Samko, S. G., & Ross, B. (1993). Integration and differentiation to a variable fractional order. Integral Transforms and Special Functions, 1(4), 277–300.

[9] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon.

[10] Xu, Y., & Zhang, H. (2017). Existence and uniqueness of solutions for variable-order fractional differential equations. Fractional Calculus and Applied Analysis, 20(5), 1243–1264.

[11] Zhang, S. (2013). Existence of solutions for two point boundary value problems with singular differential equations of variable order. Electronic Journal of Differential Equations, 245, 1–16.