Some Results of Fixed Point for Rational Expression in N- Fuzzy Metric Space with Applications

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***Abstract*—In the present paper, we proved some fixed points theorems for rational inequality in N-fuzzy metric space[which is defined by Malviya N [5]]. Some applications are also given is support of our results. Our results extend and generalize the results of Gupta.V et.al [3].and Agrawal S. et.al [1].**

***Index Terms*—N-fuzzy metric space, Rational Expression, Fixed Point, Integral type Banach contraction.**

I. Introduction

In the history of mathematical analysis there exist many generalizations of fuzzy metric spaces, In 1975, Kramosil and Michalek[4] introduced fuzzy metric space.In 1994 George and Veeramani [2] modified the concept of fuzzy metric apace and studied a Housdorff topology of fuzzy metric space. In 2010, Sun andYang[8] introduced the Q-fuzzy metric space using the definition of G-metric space.Using the concept of D*-metric space Sedghi and Shobe[7] defined M-fuzzy metric space. Recently Malviya N.[5] introduced the notion of N-fuzzy metric space usingthe concept of S- metric space and defined Pseudo N-fuzzy metric space and proved various lemmas related to topology and convergence.

In the present paper, we prove fixed point theorem using rational inequality, our result extend the results of Gupta V. et al [3] in the structure of N-fuzzy metric space and generalize the theorem of Agrawal S. et al [1]. As an application we prove integral analogue of our result.

II. Preliminaries

Definition 2.1[6]:-A map o: $[0,\ 1] \times [0,\ 1] \times [0,\ 1] \rightarrow [0,\ 1]$ is called a continuous t −norm if it satisfies the following conditions:

$$T_1: *\ (\alpha,\ 1, 1\ ) =\ \alpha, \quad *\ (0, 0, 0) =\ 0$$

$$T_2: *\ (\alpha,\ \beta, \gamma\ ) = \quad *\ (\alpha, \gamma, \beta) = *\ (\beta, \gamma, \alpha)$$

$$T_3: *\ \left(\alpha_1,\ \beta_1, \gamma_1\right) \geq \ *\ \left(\alpha_2,\ \beta_2,\ \gamma_2,\right) for\ \alpha_1 \geq \alpha_2,\ \beta_{1,} \geq \beta_{2,},\ \ \gamma_1 \geq \gamma_2$$

examples of t −norm are (1): $\alpha\ *\ \beta\ *\ \gamma = \alpha.\beta.\gamma$ and (2): $\alpha\ *\ \beta\ *\ \gamma\ =\ min\{\alpha, \beta, \gamma\}(H\ -\ type)$

Definition 2.2[5]:- A triplet $(X, N, *)$ is an N- fuzzy metric space($NFMS_s$), if X is an arbitrary (non-empty) set,* is a continuous t-norm and N is a fuzzy set on $X^3\times(0, \infty)$ satisfying the following conditions for all α, β, γ∈ X and $t, u, v > 0$:

(i) $N(\alpha, \beta, \gamma, t) > 0$

(ii) $N(\alpha, \beta, \gamma, t) = 1$ *if and only if* $\alpha = \beta = \gamma$

$(iii) N(\alpha, \beta, \gamma, u + v + t) \geq N(\alpha, \alpha, \xi, u) * N(\beta, \beta, \xi, v) * N(\gamma, \gamma, \xi, t)$ for all $\xi \in X$.

$(iv) N(\alpha, \beta, \gamma, .): (0, \infty) \to (0, 1]$ *is a continuous function.*

Example 2.1[5]: - Let $X = R$ be a real line and S be an S-metric on X defined by

$$S(\alpha, \beta, \gamma) = |\alpha - \gamma| + |\beta - \gamma|$$

$$S(\alpha, \beta, \gamma) = |\beta + \gamma - 2\alpha| + |\beta - \gamma|$$

Define $p * q * r = pqr$ *for every* $p, q, r \in [0, 1]$ and let N be the function on $X^3\times(0, \infty)$ define by $N(\alpha, \beta, \gamma, t) = \frac{t}{t+S(\alpha, \beta, \gamma)}$ for all $\alpha, \beta, \gamma \in X$ *and* t > 0.

Then $(R, N, *)$ is an N-fuzzy metric space.

Definition 2.3[5]:- Let $(X, N, *)$ is a N-fuzzy metric space then a sequence $\{\alpha_n\} \in X$ is said to be convergent to a point $x \in X$ if $N(\alpha_n, \alpha_n, \alpha, t) = 1$ for all $t > 0$.

Definition 2.4[5]:-Let$(X, N, *)$ is a N-fuzzy metric space then a sequence $\{\alpha_n\} \in X$ is called a Cauchy sequence if $x \in X$ if $N(\alpha_n, \alpha_n, \alpha_{n+p}, t) = 1$ for all $p > 0$.

Definition 2.5[5]: -Let$(X, N, *)$ be an N-fuzzy metric space. A self-map $f: X \to X$ is a fuzzy k −contraction if for all $\alpha, \beta \in X$ and for some $k \in (0, \frac{1}{3})$**, we have**

$$N(f(\alpha), f(\alpha), f(\beta), kt) \geq N(\alpha, \alpha, \beta, t)$$

Proposition 1.1[5]:-Let $(X, N, *)$ be an N-fuzzy metric space, then for all $\alpha, \beta \in X$ and $t > 0$, we have $N(\alpha, \alpha, \beta, t) = N(\beta, \beta, \alpha, t)$.

Lemma 1[5]: -For all, $\alpha, \beta \in X$, $N(\alpha, \alpha, \beta, .)$is non-decreasing.

Lemma 2[5]:-If there exist $k \in (0, \frac{1}{3})$ such that $N(\alpha, \alpha, \beta, kt) \geq N(\alpha, \alpha, \beta, t)$for all $\alpha, \beta \in X$ and $t \in (0, \infty)$, then $\alpha = \beta$.

III. Main Results

Theorem 3.1:- Let $(X, N, *)$ be a complete N-fuzzy metric space and $f: X \to X$ be a mapping satisfying

$N(\alpha, \beta, \gamma, t) = 1$ …………………………(1.1)

and

$N(f(\alpha), f(\alpha),\ f(\beta),\ kt) \geq \lambda(\alpha, \alpha, \beta,\ t)$ …………(1.2)

Where

$\lambda(\alpha, \alpha, \beta,\ t) = min\left\{\frac{N(\beta,\beta,f(\beta),t)[1+N(\alpha,\alpha,f(\alpha),t)]}{[1+N(\alpha,\alpha,\beta,t)]}, N(\alpha, \alpha, \beta, t)\right\}$……….(1.3)

For all $\alpha, \beta \in X$ and $k \in \left(0, \frac{1}{3}\right)$. then f has a fixed point.

Proof: Let us consider $\alpha \in X$ be any arbitrary point in X. Now construct a sequence $\left\{\alpha_n\right\} \in X$ such that

$f(\alpha_n) = \alpha_{n+1}$ for all $n \in N$.

Claim. $\left\{\alpha_n\right\}$ is a Cauchy sequence.

Let us take $\alpha = \alpha_{n-1}$ and $\beta = \alpha_n$ in (1.2), we get

$N\left(\alpha_n, \alpha_n,\ \alpha_{n-1},\ kt\right) = N(f(\alpha_{n-1}), f(\alpha_{n-1}),\ f(\alpha_n),\ kt) \geq \lambda(\alpha_{n-1}, \alpha_{n-1}\alpha_n,\ t)$ (1.4)

Now

$$\lambda(\alpha_{n-1}, \alpha_{n-1}, \alpha_n,\ t) = min\left\{\frac{N\left(\alpha_n,\alpha_n,f(\alpha_n),t\right)[1+N\left(\alpha_{n-1},\alpha_{n-1},f(\alpha_{n-1}),t\right)]}{[1+N\left(\alpha_{n-1},\alpha_{n-1}\alpha_n,t\right)]}, N\left(\alpha_{n-1}, \alpha_{n-1}\alpha_n, t\right)\right\}$$

$$\lambda(\alpha_{n-1}, \alpha_{n-1}, \alpha_n,\ t) = min\left\{\frac{N\left(\alpha_n,\alpha_n,\alpha_{n+1},t\right)[1+N\left(\alpha_{n-1},\alpha_{n-1}\alpha_n,t\right)]}{[1+N\left(\alpha_{n-1},\alpha_{n-1}\alpha_n,t\right)]}, N\left(\alpha_{n-1}, \alpha_{n-1}\alpha_n, t\right)\right\}$$

$$\lambda(\alpha_{n-1}, \alpha_{n-1}, \alpha_n,\ t) = min\left\{N\left(\alpha_n, \alpha_n, \alpha_{n+1}, t\right),\ N\left(\alpha_{n-1}, \alpha_{n-1}\alpha_n, t\right)\right\}$$

Now if $N\left(\alpha_n, \alpha_n, \alpha_{n+1}, t\right) \leq N\left(\alpha_{n-1}, \alpha_{n-1}\alpha_n, t\right)$ then form equation(1.4)

$$N\left(\alpha_n, \alpha_n, \alpha_{n+1}, kt\right) \geq N\left(\alpha_n, \alpha_n, \alpha_{n+1}, t\right)$$

Hence form lemma (2), our claim follows immediately. Now suppose

$$N\left(\alpha_n, \alpha_n, \alpha_{n+1}, t\right) \geq N\left(\alpha_{n-1}, \alpha_{n-1}, \alpha_n, t\right)$$

Then again from equation (1.4)

$$N\left(\alpha_n, \alpha_n, \alpha_{n+1}, kt\right) \geq N\left(\left(\alpha_{n-1}, \alpha_{n-1}, \alpha_n, t\right)\right)$$

Now by simple induction, for all n and $t > 0$, we get

$N\left(\alpha_n, \alpha_n, \alpha_{n+1}, kt\right) \geq N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{k^{n-1}}\right)$ (1.5)

By using equation (1.5), for any positive integer ‘p’we get

$$N\left(\alpha_n, \alpha_{n,}p, t\right) \geq N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) * N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) * N\left(\alpha_{n+p}, \alpha_{n+p}, \alpha_{n+1}, \frac{t}{3}\right)$$

$$N\left(\alpha_n, \alpha_n, \alpha_{n+p}, t\right) = N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) * N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) * N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+p}, \frac{t}{3}\right)$$

$$N\left(\alpha_n, \alpha_n, \alpha_{n+p}, t\right) \geq N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) * N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) * N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+2}, \frac{t}{(3)^2}\right) * N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+2}, \frac{t}{(3)^2}\right)$$

$$N\left(\alpha_n, \alpha_n, \alpha_{n+p}, t\right) = N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) * N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) * N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+2}, \frac{t}{(3)^2}\right) * N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+2}, \frac{t}{(3)^2}\right.$$

$$N\left(\alpha_n, \alpha_n, \alpha_{n+p}, t\right) = N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{k^n 3}\right) * N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{k^n 3}\right) * N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{k^{n+1}(3)^2}\right) * N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{k^{n+1}(3)^2}\right) * \ldots.$$

By the definition of fuzzy k-contraction (i.e., k<1) together with condition (1.1) and letting $n\to\infty$, we get

$$N\left(\alpha_n, \alpha_n, \alpha_{n+p}, t\right) = 1 \ *1* \ 1* 1\ldots.* 1 * 1 = 1$$

Hence, $\{\alpha_n\}$ is Cauchy sequence. Since $(X, N,*)$ is a complete $N-FMS$, there exists $\vartheta \in X$ such that

$\alpha_n = \vartheta$ (1.6)

Claim ϑ is a fixed point of f.

Consider

$$N(\vartheta, \vartheta, f(\vartheta), t) \geq N\left(\vartheta, \vartheta, f(\alpha_n), \frac{t}{3}\right) * N\left(\vartheta, \vartheta, f(\alpha_n), \frac{t}{3}\right) * N\left(f(\vartheta), f(\vartheta), f(\alpha_n), \frac{t}{3}\right) \ldots..(1.7)$$

$$= N\left(\vartheta, \vartheta, \alpha_{n_{+1}}, \frac{t}{3}\right) * N\left(\vartheta, \vartheta, \alpha_{n_{+1}}, \frac{t}{3}\right) * N\left(f\left(\alpha_n\right), f\left(\alpha_n\right), f(\vartheta), \frac{t}{3}\right)$$

$$= N\left(\vartheta, \vartheta, \alpha_{n_{+1}}, \frac{t}{3}\right) * N\left(\vartheta, \vartheta, \alpha_{n_{+1}}, \frac{t}{3}\right) * \lambda\left(\alpha_n, \alpha_n, \vartheta, \frac{t}{3k}\right)$$

Now

$$\lambda\left(\alpha_n, \alpha_n, \vartheta, \frac{t}{3k}\right) = min\left\{\frac{N\left(\vartheta,\vartheta,f(\vartheta),\frac{t}{3k}\right)[1+N\left(\alpha_n,\alpha_n,f(\alpha_n),\frac{t}{3k}\right)]}{[1+N\left(\alpha_n,\alpha_n,\vartheta,\frac{t}{3k}\right)]}, N\left(\alpha_n, \alpha_n, \vartheta, \frac{t}{3k}\right)\right\}$$

$$\lambda\left(\alpha_n, \alpha_n, \vartheta, \frac{t}{3k}\right) = min\left\{\frac{N\left(\vartheta,\vartheta,f(\vartheta),\frac{t}{3k}\right)[1+N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3k}\right)]}{[1+N\left(\alpha_n,\alpha_n,\vartheta,\frac{t}{3k}\right)]}, N\left(\alpha_n, \alpha_n, \vartheta, \frac{t}{3k}\right)\right\}$$

Taking. In above inequality and using (1.1), we get

$$\lambda\left(\vartheta, \vartheta, \vartheta, \frac{t}{3k}\right) = min\left\{N\left(\vartheta, \vartheta, f(\vartheta), \frac{t}{3k}\right), 1\right\}$$

Now if $N\left(\vartheta, \vartheta, f(\vartheta), \frac{t}{3k}\right) \geq 1$ then $\lambda\left(\vartheta, \vartheta, \vartheta, \frac{t}{3k}\right) = 1$.

Therefore from (1.7) and using definition 2.2 we get ϑ is a fixed point of f.

Now if $N\left(\vartheta, \vartheta, f(\vartheta), \frac{t}{3k}\right) \leq 1$ then $\lambda\left(\vartheta, \vartheta, \vartheta, \frac{t}{3k}\right) = N\left(\vartheta, \vartheta, f(\vartheta), \frac{t}{3k}\right)$.

Hence from equation (1.7), we get

$$N(\vartheta, \vartheta, f(\vartheta), t) \geq N\left(\vartheta, \vartheta, f(\vartheta), \frac{t}{3k}\right) \quad (1.8)$$

Since $k \in \left(0, \frac{1}{3}\right)$ therefor by lemma (2), we get $f(\vartheta) = \vartheta$.

This completes the proof of theorem 1.

Let us define $\theta = \{\frac{\phi}{\phi}: [0,1]\rightarrow[0,1]\}$ is a continuous function such that $\phi(1) = 1,\ \phi(0) = 0, \phi(a) > a$ for each $0 < a < 1\}$.

Theorem 3.2 Let $(X, N, *)$ is a N-fuzzy metric space with

$N(\alpha, \beta, \gamma, t) = 1$ (1.9)

And $f: X\rightarrow X$ be a mapping satisfying

$N(f(\alpha), f(\alpha), f(\beta), kt)\geq\phi\{\lambda(\alpha, \alpha, \beta,\ t)\}$ (1.10)

Where

$$\lambda(\alpha, \alpha, \beta,\ t) = min\left\{\frac{N(\beta,\beta,f(\beta),t)[1+N(\alpha,\alpha,f(\alpha),t)]}{[1+N(\alpha,\alpha,\beta,t)]}, N(\alpha, \alpha, \beta, t)\right\}$$

For all $\alpha, \beta\in X$ and $k\in(0, 1/3), \phi\in\theta$. Then f has a fixed point.

Proof: Since $\phi\in\theta$. This implies that $\phi(a) > a$ for each $0 < a < 1$. Thus from (1.10)

$$N(f(\alpha), f(\alpha), f(\beta), kt)\geq\phi\{\lambda(\alpha, \beta,\ t)\}\geq\lambda(\alpha, \beta,\ t)$$

Now, applying theorem 3.1, we obtain the desired result.

Applications

In this section, we gives some applications related to our results. Let us define

$\psi: [0, \infty] \rightarrow [0, \infty),\ as\ \psi(t) = \int_0^t \varphi(t)dtt > 0$, be a non- decreasing and continuous function. Moreover, for each $\epsilon > 0, \varphi(\epsilon) > 0$. Also implies that $\varphi(t) = 0$ iff $t = 0$. In the following, we prove integral analogue of theorem 3.1 in N-Fuzzy Metric Space.

Theorem 4.1 Let$(X, N, *)$ be a complete N-fuzzy metric space with

$N(\alpha, \beta, \gamma, t) = 1$ and $f: X\rightarrow X$ be a mapping satisfying

$$\int_0^{N(f(\alpha),f(\alpha),f(\beta),kt)} \varphi(t)dt\geq \int_0^{\lambda(\alpha,\alpha,\beta,\ t)} \varphi(t)dt$$

Where

$$\lambda(\alpha, \alpha, \beta,\ t) = min\left\{\frac{N(\beta,\beta,f(\beta),t)[1+N(\alpha,\alpha,f(\alpha),t)]}{[1+N(\alpha,\alpha,\beta,t)]}, N(\alpha, \alpha, \beta, t)\right\}$$

For all $\alpha, \beta\in X,\ \varphi\in\psi$ and $k\in\left(0, \frac{1}{3}\right)$. This f has a fixed point.

Proof: By taking $\varphi(t) = 1$ and applying theorem 3.1, we obtain the result.

Theorem 4.2: Let $(X, N, *)$ be a complete N-fuzzy metric space with

$N(\alpha, \beta, \gamma, t) = 1$ and $f: X\rightarrow X$ be a mapping satisfying

$$\int_0^{N(f(\alpha),f(\alpha),f(\beta),kt)} \varphi(t)dt\geq\phi\{\int_0^{\lambda(\alpha,\alpha,\beta,\ t)} \varphi(t)dt\}$$

Where

$$\lambda(\alpha, \alpha, \beta,\ t) = min\left\{\frac{N(\beta,\beta,f(\beta),t)[1+N(\alpha,\alpha,f(\beta),t)]}{[1+N(\alpha,\alpha,\beta,t)]}, N(\alpha, \alpha, \beta, t)\right\}$$

For all $\alpha, \beta\in X,\ \varphi\in\psi$ and $k\in\left(0, \frac{1}{3}\right)$. This f has a fixed point.

Proof: Since $\phi(a) > a$ for each $0 < a < 1$, therefore result follows immediately from theorem 4.1.

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