STUDY OF TOPOLOGICAL GAMES OVER SOME SPECIAL PRODUCTS AND ITS FUZZIFICATION

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***Abstract*—In this present paper, we study topological games over some special products and its fuzzification.**

***Index Terms*—Fuzzy Logic, Statistical Profiler, MATLAB.**

I. Introduction

By introducing the concept of topological game over an ideal of Hausdorff space, a game over some special product space is played. Fuzzy set theory has been applied to fuzzify some of the results obtained. Over an ideal of a topological space has played a topological game which is explained here in brief. Also by introducing the concept of rectangle in topological product spaces, some special types of products called D-Product and C-Product are studied and a game is played over such products. Lastly, it is explained how fuzzy set theory can be applied to obtain better results [1-2].

II. GAMES OVER AN IDEAL

Let G (I, X) be an infinite positional game of pursuit and evasion over I where X is a topological space and I $\subset$ P (X) s.t. (i) I am closed with respect to union (ii) I possess hereditary property [3].

Such collection I is called an ideal over X.

This game is played as follows: There are two players-P (Pursuer) and E (Evader). They choose alternately consecutive terms of a sequence < En/n$\in$N, Where N = {0,1,2,…n,….)}> of subsets of X s.t. each player knows I, E_0, E_1, ….,E_n when he is choosing E_{n+1}.

A sequence < En > of subset of X is said to be a play of the game if for all n N the following holds:

(i) $E_0 = X$ (ii) E_1, E_3, E_5, ……,E_{2n+1} are the choice of P.

(iii) E_1, E_3, E_5, ……,$E_{2n+1} \in$ I.

(iv) E_2, E_4, E_6, ……,E_{2n+2} are the choice of E.

(v) E_1, E_2, $\subset E_0$, E3, $E_4 \subset E_2$; ……,E_{2n+1}, $E_{2n+2} \subset$ E2n

(vi) $E_1 \cap E_2 = \varphi$, $E_3 \cap$ E4 = , ……..$E_{2n+1} \cap E_{2n+2} = \varphi$.

If $\cap$< E_{2n} > = then player P wins the play, otherwise Evader wins the play.

A finite sequence < E_m/ m $\leq$ n > is admissible for the game if the sequence < E_0, E_1, ……E_n, φ,φ,φ,…..φ > is a play of the game. For admissible sequence

< E_0,…..E_n> and even if s: < E_0,….,E_n > P(X) and s (<E_0,….,E_n) = E_{n+1} then s is a strategy for player P.

In case of odd n, s is said to be a strategy for evader E.

A strategy s is said to be wining for player P in the game G (I, X) if P wins each play of the game with the help of this s. Similarly, s is said to be winning for E if E wins each play of the game with the help of s [4].

We denote by P (I, X) the set of all winning strategies of P in the game G(I, X) and by E(I,X), the set of all winning strategies of E in the game G (I,X).

A topological space X is aid to be I-like if the set of all winning strategies of player is not empty i.e., if P (I, X) $\neq \varphi$.

Similarly, a space X is said to be determined, if P (I, X) $\neq\varphi$ or E (I, X) $\neq\varphi$ i.e., if X is I-like or X is anti I-like.

III. RECTANGULAR PRODUCTS

A subset A x B of a topological product space X x Y is called a rectangle. A rectangle E is said to be:

(i) Co-zero if E' & E" are co-zero in X x Y;
(ii) Zero if E' & E" are zero in X x Y;
(iii) Open if E' & E" are open in X x Y;
(iv) Closed if E' & E" are closed in X x Y;

where E' & E" are the projections of E into X and Y respectively so that E = E' x E".

A topological product X x Y is said to be strongly rectangular if each locally finite open cover of X x Y has a locally finite refinement by co-zero rectangles.

From above definitions the following conditions are seen to be equivalent:

(i) The product X x Y is strongly rectangular.
(ii) Each finite open cover of X x Y has a locally finite refinement by co-zero rectangles.
(iii) For each closed subset F and each open set U of X x Y with F U, there is a locally finite collection w by cozero rectangles s.t. F ⊂∪ W U.
(iv) X x Y is normal and for each zero-set F and each co-zero-set U of X x Y with FU, there is a locally finite collection W by co-zero rectangles such that F ⊂∪ W U.
(v) There exists a continuous map

f: X x Y [0,1] such that $f(x,y) = \sum_{tT} g_t(x) h_1(y)$

where $g_t : X\rightarrow [0,1]$ and $h_t : Y\rightarrow [0,1]$ are continuous.

IV. MODIFIED GAMES

We define the topological games G (I, X) with a slight change as follows: Each topological space considered in this paper is assumed to be a Hausdorff space. N denotes the set of all natural numbers and m denotes an infinite cardinal number. Also let $L = \{E_i \mid E_i$ is closed subsets of $X\}$.

There are two players P and E. Player P chooses a closed set Et of X with E1 L and player E chooses an open set U_1 of X with $E_1 \subset U_1$ [5].

Again, player P chooses a closed set E_2 of X with E_2 L and player E chooses an open set U_2 of X with $E_2 \subset U_2$ and so on.

The infinite sequence < E_1, U_1, E_2, U_2……> is played by G (L, X). Player P wins the play < E_1, U_2, E_2, U_2,…> if {Un : n N} covers X, otherwise player E wins.

A finite sequence < E_1, U_1,…., E_n, U_n > of subsets in X is said to be admissible for G(L,X) if the infinite sequence < E_1, U_1, …E_n, U_n, φ,φ,…..> is a play of G (L,X).

A function s is said to be a strategy for player P in G (L, X) if the domain of S consists of the void sequence φ and the finite sequence < U1, …..,Un > of open sets in X and if s (φ) ad s (U_1,……,U_n) are closed in X an belong to L.

A strategy s for player P in the game G (L,X) is said to be winning if he wins each play < E_1, U_1, E_2, U_2,….. in (L, X) such that E_1 = S (φ) and E_{n+1} = S (U_1, ….,U_n), for all n ∈ N.
We denote the following:
DL - The class of all spaces which have a discrete closed cover consisting of members of L.
FL - The class of all spaces which have a finite closed cover consisting opf members of L.
C - The class of all compact spaces.
C_m - The class of m-compact space.
I_1, I_2 - Arbitrary classes of spaces possessing hereditary property s.t.
I_1 x I_2 = {X x Y : X ∈ I_1 and Y I_2}

V. GAMES OVER SOME SPECIAL PRODUCT SPACES

Firstly, we define the following two product spaces:
D- Product: A product space X x Y is said to be a D-product if for each closed set M of X x Y and each open set O of X x Y with M ⊂ O, there is a discrete collection J by closed rectangles in X x Y such that M ⊂∪ J⊂ O.
For a closed rectangle R in X x Y, R' and R" denote the projection of R into X and Y respectively. Thus, R is a closed rectangle in X x Y if R' and R" are closed in X & Y and R is an open rectangle in X x Y if R'R" are open in X and Y such that R = R' and R" [6].
C-Product: A product space X x Y is said to be a C-product if for each closed set M of X x Y and each open set O of X x Y with M ⊂ O there is a countable collection J by closed rectangles in X x Y such that M ⊂∪ J ⊂ O.
With the help of definition of D-product, we have,
Theorem: (1) Let X and Y be spaces such that X x Y is a D-Product. If player P has winning strategies in G (l_1, X) and (l_2,Y), then he has a winning strategy in G (D (l_1 x l_2), X x Y).
Now we prove the following
Theorem: (2) Let X be a collection wise normal space and Y a subpar compact space with χ (Y) ≤ m. If player P has a winning strategy in G (DC_m, X), then every open cover of X x Y with power < m has a σ-discrete refinement by closed rectangles in X x Y.
Proof: Let s be a winning strategy of player P in G (DC_m, X). Let C be an arbitrary open cover of X x Y with |C| ≤ m.
We construct:

- (i) a sequence {J_n : n > 0} collections of closed rectangles in X x Y;
- (ii) sequence {< $\Re_n$, < ψ_n > : n ≥ 0} of the pairs of collections Rn by closed rectangles in X x Y;
- (iii) the function $\psi_n : \Re_n \rightarrow \Re_{n-1}$; satisfying the following five conditions:
- (a) J_n is σ-discrete in X x Y.
- (b) Rn is σ-discrete in X x Y.
- (c) Each F ∈ J_n is contained in some G ∈ C.
- (d) If (x,y) ∈ R_{n-1} ∈ $\Re_{n-1}$ and (x,y) ∈ ∪J_n.
  Then there is $R_n \in \Re_n$ such that (x,y) ∈ R_n and ψ_n (R_n) = R_{n-1}.
- (e) for an R ∈ $\Re_n$, Let U_n = X - R and U_k = X - (ψ_{k+1} 0…….0 ψ_n(R))', for 1 ≤ k ≤ n-1.

Then the finite sequence < E_1, U_1, ….., E_n, U_n > is admissible for G (DC_m, X).
Let J_0 = {φ} and $\Re_0$ = {X x Y}.
We suppose that the above {J_i : i < n} and {< R_i, ψ_I > : I ≤ n} are already constructed. We pick an R ∈ $\Re_n$.
Let < E_1, U_1,……, E_n, U_n > be the admissible sequence in G (DC_m, X).
Hence there is a discrete collection {Cα : α ∈ Ω (R)} by m-compact closed sets in R1 such that s(U_1,……., U_n) R' = ∪ {Cα : α ∈ Ω (R)}.
We can a choose discrete collection {Wα : α ∈ Ω (R)} of open sets in R' s.t. Cα ⊂ Wα, for all α ∈Ω (R).
Since Cα is m-compact, |C| < m, χ(Y) ≤ m and R" is subparacompact.
There is a collection $J_{n+1}^{\alpha} = \left\{ Cl\, U^{\alpha,i} \times H : i = l, \ldots\ldots, K \ and \ (k) \right\}$ and by closed rectangle in R, which satisfying the following four conditions:

(i) Each $U^{\alpha,i}$ is open in R'.

(2) $C \ \left\{U^{\alpha,i}: i = 1, \ldots\ldots, K\right\} W$.

(3) Each CI $U^{\alpha,i}$ × H_{λ} is contained in some G ∈ C.

(4) {H : ( )} is a σ-discrete closed cover of R". Then

$J^{\alpha}_{n+1}(R) = \ \{U^{\alpha}_{n+1}: \ \}$ is a -discrete in X x Y.

Put $R \ = \left\{CI\ W \ - \ \left\{U^{\alpha}_{n+1}: 1 \le i \le K\right\} \times H\right\}$,for all (k).

Again put $\overline{R} = \left(R' - \ \left\{W_{\alpha} : \ \alpha\epsilon\ (R)\right\}\right) \times R''$

Moreover, we put R_{n+1} (R) = $\{\overline{R} \cup \ \{R^{\alpha}: ()\}\ and\ \ (R)\}$.

Then R_{n+1} (R) is also σ-discrete collection by closed rectangles in R.
We set $J_{n+1} = \{J_{n+1}(R) : R \in \mathfrak{R}_n\}$ and $\mathfrak{R}_{n+1} = \cup\ \{\mathfrak{R}_{n+1}(R) : R \in \mathfrak{R}_n\}$.
The function $\psi_n: \mathfrak{R}_{n+1} \rightarrow R_n$ defined as $\psi_{n+1:}\ (\mathfrak{R}_{n+1})$ (R) = (R) for all $R \in \mathfrak{R}$.
From (a), J_{n+1} and $\mathfrak{R}_{n+1}$ are σ-discrete in X x Y.
The conditions (a) and (b) are satisfied.
By (3), the the condition (c) is also satisfied.
The conditions (d) and (e) are very clear.
Let $J = \cup\ \{J_n : n \in N\}$.
We can easily show that J is a cover of X x Y. Therefore, J is a σ-discrete refinement of C by closed rectangles in X x Y.
With the consequences of the above theorem and by assuming PC_m to be the class of all product spaces with the first factor being non-compact, the following results can be obtained easily.
(R_1) Let X be a collection wise normal space and Y and a subparacompact space with $\chi(Y) \le m$. If player P has a winning strategy in G (DCm, X), then X x Y is a D-product.
(R_2) Let X be a paracompact space and Y be a subparacompact space.
IF player P has a winning strategy is G (DC, X), then X x Y is subparacompact.
(R_3) Let X be a collection wise normal space and Y be a subparacomact space with $\chi(Y) \le m$. If player P has a winning strategy in G (DCm, X), then he has a winning strategy in G (D(PCm_m), X x Y).

VI. FUZZY GAME

A game is determined by information, decisions and goals. But human notions (ideas) and decisions are fuzzy. For, a man with immense entropy functions may err, set right and understanding a little may increases his understanding in the pursuit of some knowledge. Therefore, in a game, perfect information, decisions & goals may not be feasible. We are therefore led to the introduction of fuzzy games.

Let G = (N,v) be a nonfuzzy game of the set N = {1,2,3,…, n} of n players in which v : S → R is a real valid function (characteristic function) from a family of coalition S ⊂ N to the set of real numbers R. Hence v(A) means the gain which a coalition. A can acquire only through the action of A, the coalition A can be specified by the characteristic function τ^A as follows:

$$\tau^A(i) = \begin{cases} 1 & \text{if } I \in A; \\ 0 & \text{if } I \in A. \ / \end{cases}$$

A rate of participation τ^A (i) of a player i is defined by
τ^A (i) = 1, if a player i participates in A and
τ^A (i) = 0, if a player i does not not participate in A.
Consequently, a coalition A is represented by
= (τ^A (1), $\tau^{\wedge}$ (2),….. τ^A (n)).

A fuzzy coalition τ is defined as a coalition in which a player I can participate with a rate of participation $\tau_I \in [0,1]$ instead of {0,1}. The characteristic function nor coalitional worth function of a fuzzy game is a real valued function f: $[0,1]^n \rightarrow R$ which specifies a real number f (τ) for any fuzzy coalition τ.

This fuzzy game is denoted by FG = (N. f). By obtaining this fuzzy game, we can have the corresponding results of the previous section easily which may produce better results.

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